TEXAS INSTRUMENTS

Application Report

SNVA171–June 2006

AN-1489 Techniques of State Space Modeling

ABSTRACT

Over the past several decades many techniques have been used to model the PWM switch. These include both analytical and circuit based models that become unwieldy for fourth, and some second order systems. The simplest approach is to use the state space analytical method. This method may be used in conjunction with computational software, such as Matlab or Maple, to quickly and easily model a given power stage. In this paper state space modeling is presented in a step-by-step manner such that one may easily implement the approach in software by following a prescribed recipe.

Contents

List of Figures

1 Introduction

Over the past several decades many techniques have been used to model the PWM switch. These include both analytical and circuit based models that become unwieldy for fourth, and some second order systems. The simplest approach is to use the state space analytical method. This method may be used in conjunction with computational software, such as Matlab or Maple, to quickly and easily model a given power stage. In this paper state space modeling is presented in a step-by-step manner such that one may easily implement the approach in software by following a prescribed recipe.

2 State Space Modeling

State space modeling is a technique that describes a given system using a system of linear differential equations. These equations are easily manipulated using matrix operations and may be used to relate the internal, or state variables to the system input and output.

The state equations may be expressed in matrix form as the following:

$$\dot{x} = Ax + Bu$$
$$y = Qx + Ru \quad (1)$$

Where $\dot{x}$ is the time derivative of the state variable vector, A is the state matrix, x is the state variable vector, B is a vector, u is the input, *y* is the output, Q is a transposed vector relating the state variables to the output, and R is a vector relating the input to the output.

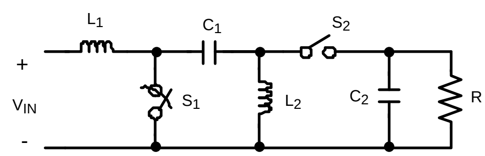

Figure 1. SEPIC Technology

Because a given network has two states in CCM , S1 on, S2 off and S1 off, S2 on, the response of the network in each state may be time weighted and averaged. For example, the SEPIC topology shown in Figure 1 may be redrawn for each of its two states using convenient shorthand for the internal variables as shown in Figure 2.

The equations for both states may be time weighted and averaged using the following relationships:

$$\dot{x} = \left[A_1 \frac{t_{ON}}{t_P} + A_2 \frac{t_{OFF}}{t_P}\right] x + \left[B_1 \frac{t_{ON}}{t_P} + B_2 \frac{t_{OFF}}{t_P}\right] u$$

$$y = \left[Q_1 \frac{t_{ON}}{t_P} + Q_2 \frac{t_{OFF}}{t_P}\right] x + \left[R_1 \frac{t_{ON}}{t_P} + R_2 \frac{t_{OFF}}{t_P}\right] u \quad (2)$$

Where the matrix subscript refers to the state of the network.

Alternatively, these relationships may be expressed as the following:

$$\dot{x} = [A_1 d + A_2 d'] x + [B_1 d + B_2 \text{ x } d'] u$$

$$y = [Q_1 d + Q_2 d'] x + [R_1 d + R_2 d'] u$$

Where

$$d = \frac{t_{ON}}{t_p} \quad (3)$$

$$d' = 1 - d = \frac{t_{OFF}}{t_p} \quad (4)$$

The variables x, d, u, and y have both large and small signal components. Each variable in relation to its components may be expressed as:

$$x = X + \chi \quad (5)$$
$$d = D + \delta \quad (6)$$
$$u = V_{IN} + v_{in} \quad (7)$$
$$y = V_{OUT} + v_{out} \quad (8)$$

Where the first and second terms on the right hand side of the equality correspond to the large and small signal components of a given variable.

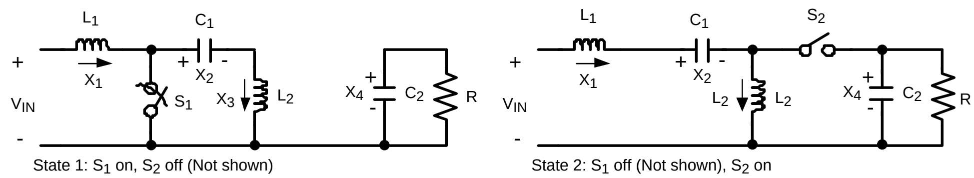

Figure 2. Network states 't_{on}' and 't_{off}'

Substituting these expressions into the time weighted average expression results in the following:

$$\dot{x} = [A_1 (D+\delta) + A_2 (1-D-\delta)](X + \chi) + [B_1 (D+\delta)$$
$$+ B_2 (1-D-\delta)](V_{IN} + v_{in})$$

$$V_{OUT} + v_{out} = [Q_1 (D+\delta) + Q_2 (1-D-\delta)](X + x\chi) + [R_1 (D+\delta) + R_2 (1-D-\delta)](V_{IN} + v_{in})$$

Because χ, δ, v_{in}, and v_{out} are small signals, the product of two small signals will have a negligible effect on the system response. These second order terms may be ignored without impairing the result. It is also important to note that, depending on the variables of interest, assumptions may be made that change each expression. This is described in the following sections.

3 Large Signal Relationship

In order to determine the large signal input-to-output relationship the small signal perturbations are considered to be negligible and set equal to zero. Doing so, the large signal response may be expressed as:

$$0 = [A_1D + A_2D1']X + [B_1D + B_2D'] V_{IN}$$

$$V_{OUT} = [Q_1D + Q_2D']X$$

$$\frac{V_{OUT}}{V_{IN}} = -\left[Q_1 \; D + Q_2 \; D'\right]\left[A_1 \; D + A_2 \; D'\right]^{-1}\left[B_1 \; D + B_2 \; D'\right] \tag{9}$$

4 Small Signal Relationship

Determining the small signal control-to-output transfer function is analogous to the large signal case. This is to say that perturbations on the input (vin) are ignored. Doing so, the relationships used to determine the small signal control-to-output transfer function may now be expressed as:

$$\dot{x} = [A_1D + A_2D']X + [B_1D + B_2D']$$

$$V_{IN} + [(A_1 - A_2) + (B_1 - B_2)\; V_{IN}]\; \delta$$

$$V_{OUT} + v_{out} = [Q_1D + Q_2 \times D']X + [Q_1D + Q_2 \times D']\; \chi + [(Q_1 - Q_2)X]\; \delta + [R_1(D + \delta) + R_2(1\text{-}D\text{-}\delta)]V_{IN}$$

To solve for the small signal control-to-output transfer function, the equations above must be converted to the frequency domain using the Laplace transform. The corresponding small signal control-to-output response may be expressed as:

$$\chi(s) = [sI - (A_1 \times D + A_2D')]^{-1}\; [(A_1 - A_2)X + (B_1 - B_2)\; VIN]\; x\delta(s)$$

The small signal control-to-output transfer function may be expressed as:

$$\frac{V_{OUT}}{\delta}(s) = \left[Q_1 \; D + Q_2 \; D'\right]\left[s \; I - (A_1 \; D + A_2 \; D')\right]^{-1} \left[(A_1 - A_2)X + (B_1 - B_2) \; V_{IN}\right] + (Q_1 - Q_2)X \tag{10}$$

Determining the small signal line-to-output transfer function is analogous to the small signal control-to-output case. This is to say that perturbations on the duty cycle (δ) are ignored. Doing so, the small signal line-to-output transfer function may be expressed as:

$$\frac{V_{OUT}}{V_{IN}}(s) = \left[Q_1 \; D + Q_2 \; D'\right]\left[s \; I - (A_1 \; D + A_2 \; D')\right]^{-1}B \tag{11}$$

5 Application to the SEPIC Power Stage

State equations for the SEPIC power stage, shown in Figure 2, in State 1 may be expressed as the following:

$$\dot{x}_1 = \frac{V_{IN}}{L_1}$$

$$\dot{x}_2 = \frac{x_3}{C_1}$$

$$\dot{x}_3 = \frac{x_2}{L_2}$$

$$V_{OUT} = x_4 \tag{12}$$

In matrix form this is expressed as the following:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{C_1} & 0 \\ 0 & -\frac{1}{L_2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{R\,C_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} \frac{1}{L_1} \\ 0 \\ 0 \\ 0 \end{bmatrix} V_{IN} \tag{13}$$

$$V_{OUT} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{14}$$

By induction it becomes apparent that the matrices A_1, B_1, Q_1, and R_1 may be expressed as:

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{C_1} & 0 \\ 0 & -\frac{1}{L_2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{RC_2} \end{bmatrix} \tag{15}$$

$$B_1 = \begin{bmatrix} \frac{1}{L_1} \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{16}$$

$$Q_1 = [0\ 0\ 0\ 1] \tag{17}$$

$R_1 = [0\ 0\ 0\ 0]$ (18)

The state equations of the network, shown in Figure 2, in State 2 may be expressed as the following:

$$\dot{x}_1 = -\frac{x_2}{L_1} - \frac{x_4}{L_1} + \frac{V_{IN}}{L_1}$$

$$\dot{x}_2 = \frac{x_1}{C_1}$$

$$\dot{x}_3 = \frac{x_4}{L_2}$$

$$\dot{x}_4 = \frac{x_1}{C_2} - \frac{x_3}{C_2} - \frac{x_4}{R\ C_2}$$

$$V_{OUT} = x_4$$

(19)

In matrix form this is expressed as the following:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{L_1} & 0 & -\frac{1}{L_1} \\ \frac{1}{C_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{L_2} \\ \frac{1}{C_2} & 0 & -\frac{1}{C_2} & -\frac{1}{R\ C_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} \frac{1}{L_1} \\ 0 \\ 0 \\ 0 \end{bmatrix} V_{IN}$$

(20)

$$V_{OUT} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

(21)

By induction it becomes apparent that the matrices A_2, B_2, Q_2, and R_2 may be expressed as:

$$A_2 = \begin{bmatrix} 0 & -\frac{1}{L_1} & 0 & -\frac{1}{L_1} \\ \frac{1}{C_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{L_2} \\ \frac{1}{C_2} & 0 & -\frac{1}{C_2} & -\frac{1}{RC_2} \end{bmatrix}$$

$$B_2 = \begin{bmatrix} \frac{1}{L_1} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(22)

$Q_2 = [0\ 0\ 0\ 1]$ (23)

$R_2 = [0\ 0\ 0\ 0]$ (24)

The large signal input-to-output relationship and associated large signal state relationships may be expressed as:

$$\frac{V_{OUT}}{V_{IN}} = \frac{D}{D'}$$

$$x_1 = \frac{D^2 \; V_{IN}}{D'^2 \; R}$$

$$x_2 = V_{IN}$$

$$x_3 = \frac{D \; V_{IN}}{-D' \; R}$$

$$x_4 = \frac{D \; V_{IN}}{D'} \tag{25}$$

The small signal control-to-output relationship may be expressed as:

$$\frac{V_{OUT}}{\delta}(s) = \frac{A_1 s^3 + A_2 s^2 + A_3 s + A_4}{A_5 s^4 + A_6 s^3 + A_7 s^2 + A_8 s + A_9} V_{IN} \tag{26}$$

$$A_1 = -L_1C_1L_2D \tag{27}$$

$$A_2 = L_1C_1RD'^2 \tag{28}$$

$$A_3 = -D^2L_1 \tag{29}$$

$$A_4 = D'^2R \tag{30}$$

$$A_5 = D'^2L_1C_1L_2C_2R \tag{31}$$

$$A_6 = D'^2L_1C_1L_2 \tag{32}$$

$$A_7 = D'^2R(L_1C_1D'^2 + L_2C_2D'^2 + C_1L_2D'^2 + L_1C_2D^2 \tag{33}$$

$$A_8 = D'^2 (L_2D'^2 + L_1D^2) \tag{34}$$

$$A_9 = D'^4R \tag{35}$$

The small signal line-to-output transfer function may be expressed as:

$$\frac{V_{OUT}}{V_{IN}}(s) = \frac{A_1 s^2 + A_2}{A_3 s^4 + A_4 s^3 + A_5 s^2 + A_6 s + A_7} \tag{36}$$

$$A_1 = C_1L_2RD' \tag{37}$$

$$A_2 = RDD' \tag{38}$$

$$A_3 = L_1C_1L_2C_2R \tag{39}$$

$$A_4 = L_1C_1L_2 \tag{40}$$

$$A_5 = R (L_1C_1D'^2 + L_2C_2D'^2 + L_1C_2D^2 + C_1L_2D'^2) \tag{41}$$

$$A_6 = L_1D^2 + L_2D'^2 \tag{42}$$

$$A_7 = RD'^2 \tag{43}$$

While the small signal expressions above provide little insight into the contribution of each component to the response of the power stage, this is of little or no consequence. Components in the power stage are typically selected according to large signal, DC, requirements of the system. The small signal, AC, response is simply an artifact of component selection.

6 Using the Model

To take full advantage of this model, one must find the poles and zeros using numerical techniques. The transfer function poles and zeros are easily calculated using routines in computational software. Once the pole and zero locations of the power stage are found, the switcher can be compensated in the usual manner.

7 Conclusion

It should become apparent that the state space method of deriving equations for any given variable in terms of another variable, large signal or small signal, becomes manageable using elementary matrix operations. The necessary matrix and numerical operations are facilitated by the use of computational software. Overall this process allows one to quickly and easily derive the transfer functions and determine the response of a given network.

IMPORTANT NOTICE

Texas Instruments Incorporated and its subsidiaries (TI) reserve the right to make corrections, enhancements, improvements and other changes to its semiconductor products and services per JESD46, latest issue, and to discontinue any product or service per JESD48, latest issue. Buyers should obtain the latest relevant information before placing orders and should verify that such information is current and complete. All semiconductor products (also referred to herein as “components”) are sold subject to TI’s terms and conditions of sale supplied at the time of order acknowledgment.

TI warrants performance of its components to the specifications applicable at the time of sale, in accordance with the warranty in TI’s terms and conditions of sale of semiconductor products. Testing and other quality control techniques are used to the extent TI deems necessary to support this warranty. Except where mandated by applicable law, testing of all parameters of each component is not necessarily performed.

TI assumes no liability for applications assistance or the design of Buyers’ products. Buyers are responsible for their products and applications using TI components. To minimize the risks associated with Buyers’ products and applications, Buyers should provide adequate design and operating safeguards.

TI does not warrant or represent that any license, either express or implied, is granted under any patent right, copyright, mask work right, or other intellectual property right relating to any combination, machine, or process in which TI components or services are used. Information published by TI regarding third-party products or services does not constitute a license to use such products or services or a warranty or endorsement thereof. Use of such information may require a license from a third party under the patents or other intellectual property of the third party, or a license from TI under the patents or other intellectual property of TI.

Reproduction of significant portions of TI information in TI data books or data sheets is permissible only if reproduction is without alteration and is accompanied by all associated warranties, conditions, limitations, and notices. TI is not responsible or liable for such altered documentation. Information of third parties may be subject to additional restrictions.

Resale of TI components or services with statements different from or beyond the parameters stated by TI for that component or service voids all express and any implied warranties for the associated TI component or service and is an unfair and deceptive business practice. TI is not responsible or liable for any such statements.

Buyer acknowledges and agrees that it is solely responsible for compliance with all legal, regulatory and safety-related requirements concerning its products, and any use of TI components in its applications, notwithstanding any applications-related information or support that may be provided by TI. Buyer represents and agrees that it has all the necessary expertise to create and implement safeguards which anticipate dangerous consequences of failures, monitor failures and their consequences, lessen the likelihood of failures that might cause harm and take appropriate remedial actions. Buyer will fully indemnify TI and its representatives against any damages arising out of the use of any TI components in safety-critical applications.

In some cases, TI components may be promoted specifically to facilitate safety-related applications. With such components, TI’s goal is to help enable customers to design and create their own end-product solutions that meet applicable functional safety standards and requirements. Nonetheless, such components are subject to these terms.

No TI components are authorized for use in FDA Class III (or similar life-critical medical equipment) unless authorized officers of the parties have executed a special agreement specifically governing such use.

Only those TI components which TI has specifically designated as military grade or “enhanced plastic” are designed and intended for use in military/aerospace applications or environments. Buyer acknowledges and agrees that any military or aerospace use of TI components which have ***not*** been so designated is solely at the Buyer's risk, and that Buyer is solely responsible for compliance with all legal and regulatory requirements in connection with such use.

TI has specifically designated certain components which meet ISO/TS16949 requirements, mainly for automotive use. Components which have not been so designated are neither designed nor intended for automotive use; and TI will not be responsible for any failure of such components to meet such requirements.

Products		**Applications**	
Audio	www.ti.com/audio	Automotive and Transportation	www.ti.com/automotive
Amplifiers	amplifier.ti.com	Communications and Telecom	www.ti.com/communications
Data Converters	dataconverter.ti.com	Computers and Peripherals	www.ti.com/computers
DLP® Products	www.dlp.com	Consumer Electronics	www.ti.com/consumer-apps
DSP	dsp.ti.com	Energy and Lighting	www.ti.com/energy
Clocks and Timers	www.ti.com/clocks	Industrial	www.ti.com/industrial
Interface	interface.ti.com	Medical	www.ti.com/medical
Logic	logic.ti.com	Security	www.ti.com/security
Power Mgmt	power.ti.com	Space, Avionics and Defense	www.ti.com/space-avionics-defense
Microcontrollers	microcontroller.ti.com	Video and Imaging	www.ti.com/video
RFID	www.ti-rfid.com		
OMAP Applications Processors	www.ti.com/omap	**TI E2E Community**	e2e.ti.com
Wireless Connectivity	www.ti.com/wirelessconnectivity		