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A Simplified Approach to dc Motor Modeling for Dynamic Stability Analysis

Edited by Mickey McClure *Power Supply Control Products*

ABSTRACT

When we say that an electric motor is a device that transforms electric power into mechanical power, we say two things. First, that the motor is – and behaves as – a transformer. Second, that it stands at the dividing line between electrical and mechanical phenomena. In the case of permanent magnet (PM) motors, this power transformation works in both directions so that the elecrical impedance depends on the mechanical load, while the mechanical behavior of the motor depends on the conditions at the electrical end. This being the case, it should be possible to represent a motor's mechanical load, on the electrical side, by a set of familiar electrical components such as capacitors or resistors.

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1 Choosing a Unit System

Consider for a moment the system of measurement units that has been chosen.

The metric system of units has undergone a number of changes in its history, of which the latest is the SI (Systeme International d'Unites). This system has become popular in most of the industrialized world, largely because it is a coherent system, in which the product or quotient of two or more units is the unit of the resulting quantity. Certain simplifications result from using this form of the metric system.

In the SI system, force IS measured in Newtons (N) and distance in meters (m). Consequently, the units of torque are Nm (see Table 1.). If a motor shaft rotates at an angular velocity of ω_M radians per second, with torque T_M, the mechanical power output will be equal to the product T_M, and ω_M and the units will be watts if T_M is in Nm.

Motor manufacturers usually specify a torque constant (K_T) and a voltage constant (K_V) for their motors. These constants have different values when the torque and speed are measured in English units, but they have the same numerical value when SI units are used. This becomes obvious when you consider that the electrical input power must be equal to the mechanical output power:

$$V_A I_A = T_A \omega_M \text{ (Watts)} \tag{1}$$

$$\frac{V_A}{\omega_M} = \frac{T_M}{I_A} = K_{TV} \tag{2}$$

where V_A is the internally generated armature voltage, or back emf. and IA IS the armature current. (See Figure 1–1 for definition of motor terms.)

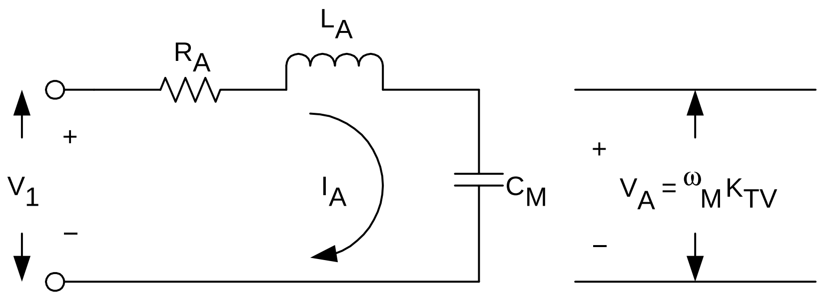
Table 1. Units Conversion

THESE UNITS	$\left\{\begin{matrix} \times \leftarrow = \\ = \rightarrow - \end{matrix}\right\}$	SI UNITS	DIM
oz	2.78×10^{-1}	N	MLT^{-2}
lb	4.448	N	MLT^{-2}
in	2.54×10^{-2}	m	L
ft	$3.048 \times 10–1$	m	L
gf	9.807×10^{-3}	N	MLT^{-2}
g cm^2	10–7	Nm sec^2	ML^2
ft lb sec^2	1.356	Nm sec^2	ML^2
oz in sec^2	7.063×10^{-3}	Nm sec^2	ML^2
ft lb	1.356	Nm	$ML2T^{-2}$
oz in	7.063×10^{-3}	Nm	$ML2T^{-2}$

NOTE: The dimensions are M (mass), L (length), and T (time). The gram (g) is a unit of mass and the gram-force (gf) is a unit of force. The pound (lb) and the ounce (oz) are included as units of force only.

Applying the the same principle to the familiar electrical transformer yields the turns ratio:

Figure 1 shows this series RLC circuit is an excellent model of a dc motor loaded with an essentially inertial load. Here, J is the total moment of inertia, including the rotor's J_M.



ω_M = motor speed in rad/sec

K_{TV} in Nm/A or V seconds

$C_M = \frac{J}{K_{TV}}$ (Farads)

Figure 1. RLC Circuit

$$V_1 I_1 = V_2 I_2 \text{ (Watts)} \tag{3}$$

$$\frac{V_1}{V_2} = \frac{I_2}{I_1} = \frac{N_1}{N_2} \tag{4}$$

Thus, the non-dimensional turns ratio N1/N2 is analogous to the dimensional torque (or voltage) constant K_{TV}. Furthermore, equations (2) and (4) give us a clear hint that the angular velocity (ω_M) is analogous to voltage, while the torque (T_M) is analogous to current.

The units of Km may be either Nm/A. or Vsec/rad. Thus, specifying both K_T and K_V for a motor is like measuring and specifying both the voltage ratio and the current ratio of a transformer, and can only make sense where redundancy is required.

2 The Motor as a Transformer

Analogies have been established K_{TV} and a transformer’s turns ratio; between angular velocity and voltage; and between torque and current. If the motor behaves as a transformer, then there is the expectation to find the square of K_{TV} involved in something analogous to impedance transformation.

If a constant current I_A is applied to the armature of a motor whose load is its own moment of inertia J_M (Nm sec^2). According to Newton’s law for rotating objects,

$$T_M = N_M \alpha_M \tag{5}$$

where α_M is the angular acceleration $d\omega_M/dt$

Since $T_M = I_A \times K_{TV}$ (as shown in equation 2)

$$I_A \; K_{TV} = J_M \times \frac{d\omega_M}{dt} \tag{6}$$

Furthermore, (also from equation 2),

$$\omega_M = \frac{V_A}{K_{TV}} \tag{7}$$

so that

$$I_A = \frac{J_M}{K_{TV}} \times \frac{dV_A}{dt} \tag{8}$$

Equation 6 has a familiar form, and yields the recognizable quantity $J_M/K^2{}_{TV}$ as a capacitor. It follows that the motor *reflects* a moment of inertia, J_M, back to the electrical primary as a capacitor of $J_M/K^2{}_{TV}$ Farads. To check this result, equate the energy stored kinetically in J_M, with the electrical energy stored in a capacitor C_M:

$$\frac{1}{2}\, C_M V^2{}_A = \frac{1}{2}\, J_M \omega^2{}_A \tag{9}$$

$$C_M = J_M \left(\frac{\omega_M}{V_A}\right)^2 \tag{10}$$

since $\frac{\omega_M}{V_A} = \frac{1}{K_{TV}}$,

$$C_M = \frac{J_M}{K^2{}_{TV}} \text{ Farads} \tag{11}$$

Similarly, a torsional spring with spring constant K_S (Nm/rad) is reflected as an inductance of $K^2{}_{TV}$ / K_S Henries. And a viscous damping component B (Nm sec/rad) appears as a resistor of $K^2{}_{TV}$ / B Ohms.

3 A Motor Model

Once mechanical load can be represented by means of electric elements, an equivalent circuit of the motor and its mechanical load can be drawn. The armature has a finite resistance R_A and an inductance L_A, through which the torque-generating current I_A must flow. These components are not negligible and must be included. An inertially loaded motor can be represented as in Figure 1, where the moment of inertia J, is the sum of the load's J_L and the rotor's J_M.

In practice, the moment of inertia that the motor must work against (or with, depending on perspective) is by far the most important component of the mechanical load. A frictional component also exists, to be sure, but because it is largely independent of speed, it would be represented electrically as a constant current source, which could not affect the dynamic behavior of the motor. And since a torsional spring which would affect motor behavior is rarely found in practice, only the inertial problem will be discussed.

3.1 Measuring the Components

The measurement of R_A and L_A is not difficult. A good ohmmeter will get you R_A, and you can measure the electrical time constant τ_E to calculate L_A:

$$L_A = \tau_E R_A \tag{12}$$

Just make sure that the rotor remains stationary during these measurements.

In order to determine the value of the capacitor, C_M, the shaft speed needs to be quantified. If the motor being measured is a brushless dc motor, the signal from one of the Hall effect devices can be used as a tachometer. If the Hall frequency is f_H and the number of rotor poles is P, angular velocity ω_M is:

$$\omega_A = \frac{4\pi f_H}{P}\ \text{(rad/sec)} \tag{13}$$

A strobe-light or some other means to measure speed is required with other types of motors.

C_M can be measured efficiently by using the mechanical time constant T_M, by driving the motor with a constant voltage driver and measuring the time it takes to accelerate from zero speed to 63% of the highest speed achievable at the voltage used. To set a safe limit to the starting current, reduce the supply voltage or add a serves resistor with the motor, or both. The set-up is shown in Figure 2. Note that the armature resistance R_A is already known, and resistors (R_B) can be added if needed, to limit the armature current I_A to a value that is safe for both driver and motor.

Initially, allow the motor run freely and measure ω_{MAX} and I_{MAX}, and use these values to calculate the armature voltage V_{MAX}.

$$V_{MAX} = V_{CC} - V_{SAT} - I_{MAX}\left(R_A + R_B\right) \tag{14}$$

Figure 2 shows the set-up for measurement of $C_M = J/K_{TV}$ of a 3-phase brushless dc motor with inertial load J1. The motor voltage $V_M = V_{CC} - V_{SAT}$, where V_{SAT} is the output saturation voltage.

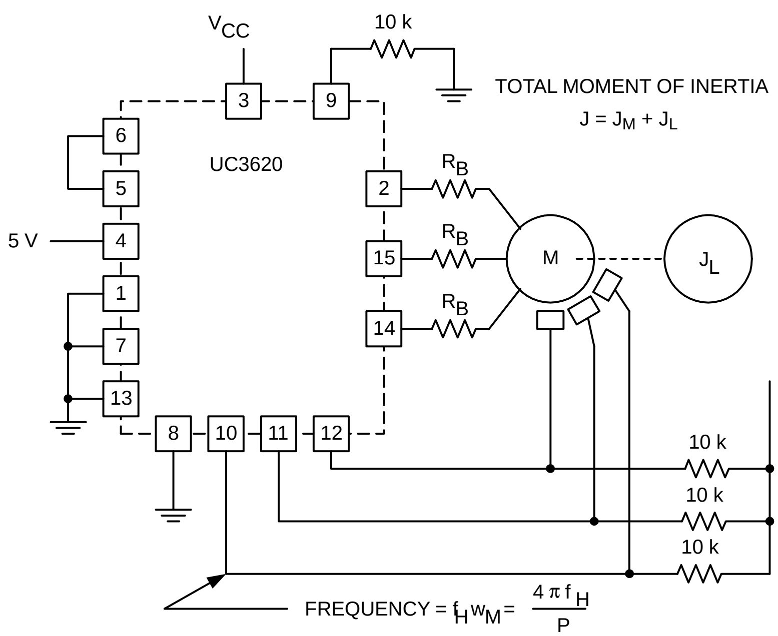


Figure 2. 3-phase Brushless dc Motor with Inertial Load

Here V_{CC} is the supply voltage, V_{SAT} is the saturation voltage of the driving circuit, and I_{MAX} is the current drawn by the unloaded motor at maximum speed.

$$K_{TV} = \frac{V_{MAX}}{\omega_{MAX}}\ \text{(Vsec/rad)} \tag{15}$$

Next, set the oscilloscope time scale to that a Hall frequency equal to 63% of w_{MAX}.

$$\omega_M = 0.63 \times \omega_{MAX} \tag{16}$$

By holding and releasing the motor shaft, take several readings of the time T_M required to accelerate from zero to ω_M. Remember that these readings are taken *on the fly*, since the motor continues to accelerate towards the maximum speed ω_{MAX}. Having obtained a good value of T_M, calculate C_M:

$$C_M = \frac{T_M}{\left(R_A + R_B\right)} \text{ Farads} \tag{17}$$

This completes the RLC equivalent circuit. If the value of J_M is also required, it too can be calculated:

$$J_M = C_M \, K^2{}_{TV} \tag{18}$$

3.2 The Motor's Transfer Function

In the circuit of Figure 1, V is the voltage applied to the motor leads and V_A is the actual armature voltage, or back-EMF. This latter voltage is equal to $\omega_M K_{TV}$, so that if it is necessary to derive an expression relating the speed to the applied voltage, it can be written:

$$\frac{\omega_M}{V_1} = \frac{1}{K_{TV}} \times \frac{V_A}{V_1} \ \text{(rad/Vsec)} \tag{19}$$

If V_1 is a constant voltage, the speed Q, will also be constant. This is clear from the circuit of Figure 1 as well as from the author's experience with motors. If however, V_1 varies sinusoidally at some frequency f, the speed ω_M will vary similarly, but the amplitude and phase will in general be different from those of the driving function. This fact is very important if the motor is included in a feedback loop, because the motor's contribution to the overall loop gain and phase shift is necessary to determine stability. The motor's transfer function (i.e. equation 19 expressed as a function of frequency) yields a precise description of how the amplitude and phase behave at different frequencies. To do this, use the variable jω, where $j = \sqrt{-1}$ and $\omega = 2\,\pi\,f$.

$$\frac{V_A(j\omega)}{V_1(j\omega)} = \frac{\left(j\omega C_m\right)^{-1}}{\left(j\omega^2\right) L_A\, C_M + j\omega R_A\, C_M + 1} \tag{20}$$

$$\frac{V_A(j\omega)}{V_1(j\omega)} = \frac{1}{\left(j\omega^2\right) L_A\, C_M + j\omega R_A\, C_M + 1} \tag{21}$$

$$L_A C_M = \frac{1}{\omega^2{}_n} \tag{22}$$

where ω_n, is the natural frequency of the circuit. since the circuit.

$$R_A C_M = \frac{R_A C_M\, L_A}{L_A} = \frac{R_A}{\omega^2{}_n L_A} = \frac{1}{Q\omega_n} \tag{23}$$

since the circuit Q is:

$$Q = \frac{\omega^2{}_n L_A}{R_A}$$

Therefore,

$$\frac{V_A(j\omega)}{V_1(j\omega)} = \frac{1}{K_{tv}} \times \frac{1}{\left(\frac{j\omega}{\omega_n}\right)^2 + \frac{j\omega}{Q\omega_n} + 1} \tag{24}$$

Furthermore, using equation 19,

$$\frac{\omega_M(j\omega)}{10} V_1(j\omega) = \frac{1}{K_{tv}} \times \frac{1}{\left(\frac{j\omega}{\omega_n}\right)^2 + \frac{j\omega}{Q\omega_n} + 1} \tag{25}$$

Since the values of K_{TV}, ω, and Q are known, the magnitude and phase angle of equation 25 for various values of $j\omega$ can be calculated. For a given $\omega = \omega_1$, equation 25 can be evaluated into a complex number $A_1 + jB_1$, whose angle is,

$$e_1 = \tan^{-1} \times \frac{B_1}{A_1} \tag{26}$$

and whose magnitude can be expressed in decibels as follows:

$$M_1 = 20 \log_{10} \sqrt{\left(A_1\right)^2 + \left(B_1\right)^2} \tag{27}$$

A plot of these quantities, using a logarithmic frequency scale, is called a Bode plot, and can be a handy tool in understanding how the device will affect the final loop performance.

4 A Disk Drive Example

A small three phase brushless dc motor, measured as above, has the following characteristics:

- K_{TV} = 0.015 Nm/A, or Vsec/rad
- R_A = 2.5 Ω
- L_A = 0.002 Hy
- J = 0.001 Nm sec

The J value was measured with three magnetic discs mounted, and represents the actual value required for the application. Using equation 11.

$$C_M = \frac{J}{\left(K_{TV}\right)^2} = \frac{.001}{(.015)^2} = 4.44 \text{ fd} \tag{28}$$

This may seem like an unusually large value for a capacitor, but it simply reflects the large amounts of kinetic energy that can be stored in the included inertia.

$$\omega_M = \frac{1}{\sqrt{L_A \times C_M}} = \frac{1}{\sqrt{0.002 \times 4.44}} = 10.61\ \text{rad/sec} \tag{29}$$

From equation 22:

$$Q = \frac{1}{\omega_n \times L_A} = \frac{10.61 \times 0.002}{2.5} = 0.0085 \tag{30}$$

From equation 23:

$$\frac{\omega_M \times (j\omega)}{V_1(j\omega)} = \frac{6.667}{\left(\frac{j\omega}{10.61}\right)^2 + \frac{j\omega}{0.09} + 1}\ \text{rad/Vsec} \tag{31}$$

A calculator that is pre-programmed to operate with complex numbers (e.g. HP-28C, or HP-15C) makes the evaluation of this equation an easy task. With the HP-28C you can set up a USER routine called BODE, as follows:

<<DEG DUP ABS LOG 20 X SWAP ARG>>

This will convert a complex number x + jy into $20 \log \sqrt{x^2 + y^2}$ at level 2, and arc tan (y/x) at level 1. Table 2 shows a list of several such computations of equation 31:

At ω = 0, the gain is simply 66.67 rad/Vsec. As ω increases from zero, the gain decreases as shown in the GAIN column of Table 2. In a Bode plot, the gain relative to the initial, or dc, gain is shown. Therefore, subtracting 66.67db from each gain value in Table 2 results in a plot shown in Figure 3. This is the same as plotting only the function (32) which can be compared with equation 31.

Table 2. Calculated Values of Equation 31

rad/sec	$\frac{\omega_M (j\omega)}{V_1(j\omega)}$	Gain (db)	Phase (degrees)
0.01	65.9 – j 7.32	36.4	–6.3
0.03	60 – j 20	36.0	–18.4
0.1	29.8 – j 33.2	33.0	–48.0
0.3	5.5 j 18.4	25.7	–73.3
1.0	0.53 – j 5.95	15.5	–84.9
3.0	0.06 – j 2.00	6.0	–88.4
10.0	0 – j 0.60	–4.4	–89.9
20.0	–4.2 x 103 – j 0.20	–14.0	–91.2
100	–4.7 x 103 – j 0.06	–24.5	–94.5
300	–4.5 x 103 – j 0.02	–34.2	–103.5
1000	–2.9 x 103 – j 3.7 x 10–3	–46.6	–128.6
3000	–7.1 x 103 – j 3 x 10–4	–62.3	–157.4

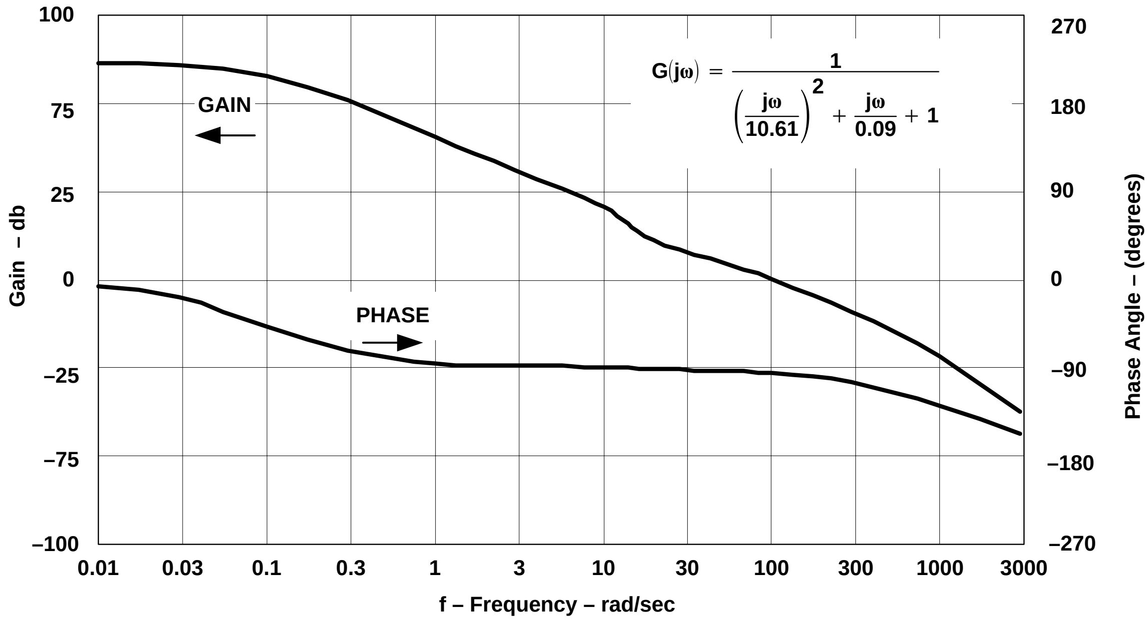


Figure 3. Bode Plot of Motor Data in Example

Note that up to about 100 rad/sec (15.9 Hz) the phase lag barely exceeds 90 degrees. The first pole occurs at ω = 0.09 rad/sec, at which point the phase lag IS 45 degrees. The second pole, widely separated from the first in this case, occurs at a frequency in excess of 1000 rad/sec, as seen in the further bend in the phase curve. The gain, which is drooping at a rate of –20 db per decade below 100 rad/sec, now begins to bend towards a steeper droop of 40 db/decade after the second pole is reached. At very high frequencies, the phase lag will reach 180 degrees.

Used in a speed control feedback loop, this motor will perform well, provided that the user takes this gain and phase behavior into account. This is done by incorporating the motor transfer function into the overall loop equation, which will include other components. The user's understanding the motor's behavior improves with this type of analysis, which makes comparisons between different motors more clear and articulate.

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Mailing Address:

Texas Instruments
Post Office Box 655303
Dallas, Texas 75265