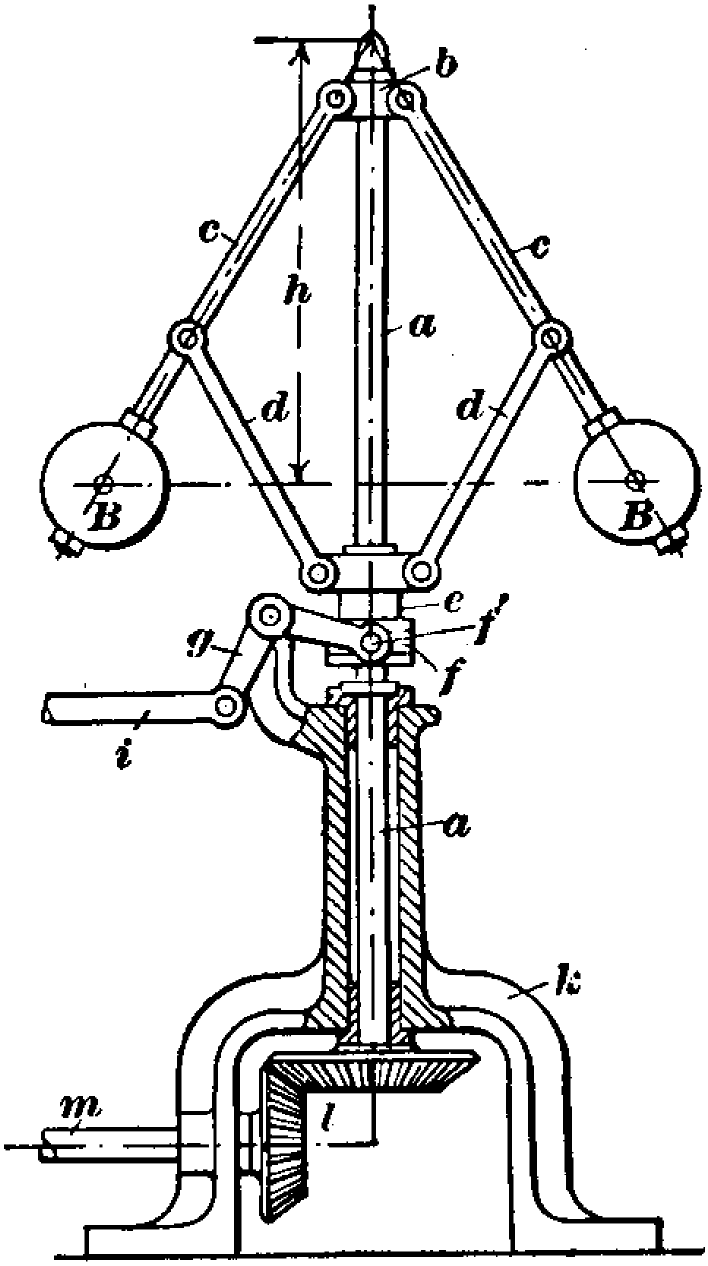
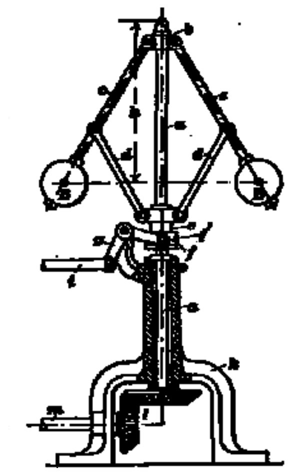
Control Theory Workshop

Student Manual



Control Theory Workshop



TEXAS INSTRUMENTS

ver. 12

Agenda

1. Fundamental Concepts
- *Linear systems*
- *Transient response classification*
- *Frequency domain descriptions*

Tutorial 1.1: Linear systems
Tutorial 1.2: Second order response

2. Feedback Control
- *Effects of feedback*
- *Steady state error*
- *Stability & bandwidth*

3. Controller Design
- *Phase compensation*
- *Root locus analysis*
- *Transient tuning*

Tutorial 3.1: Phase lead compensator design
Tutorial 3.2: Buck control example
Tutorial 3.3: Root locus design exercise
Tutorial 3.4: PID controller tuning

4. Discrete Time Systems
- *Sampled systems*
- *The z-transform*
- *z plane mapping*

5. Digital Control Design
- *Pole-zero matching*
- *Numerical approximation*
- *Invariant methods*
- *Direct digital design*

Tutorial 5.1: Matched pole-zero example
Tutorial 5.2: Discrete conversion comparison
Tutorial 5.3: Direct digital design exercise

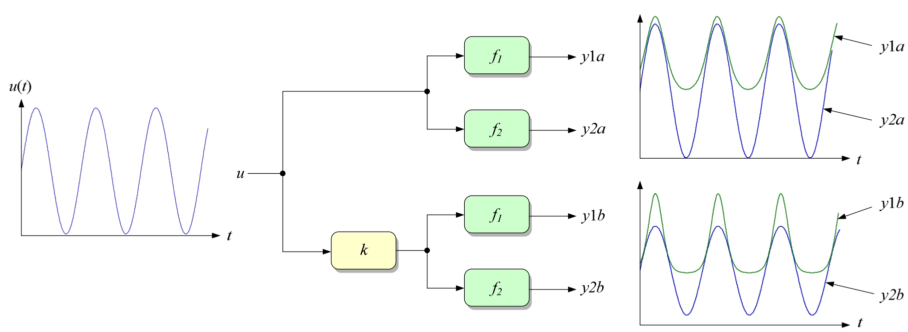
6. Digital Control Systems Implementation
- *Sample rate selection*
- *Sample to output delay*
- *Reconstruction*
- *Control law implementation*
- *Aliasing*

1. Fundamental Concepts

- **Linear systems**
- **Transient response classification**
- **Frequency domain descriptions**

Linearity

- For a **linear** system, if a scale factor is applied to the input, the output is scaled by the same amount.



- This is the **homogenous** property of a linear system $\boxed{f(k\,u) = k\,f(u)}$

- The **additive** property of a linear system is $\boxed{f(u_1 + u_2) = f(u_1) + f(u_2)}$

Terminology of Linear Systems

- Homogenous and additive properties combine to form the principle of **superposition**, which all linear systems obey

$$f(k_1u_1 + k_2u_2) = k_1 f(u_1) + k_2 f(u_2)$$

- The ordinary differential equation with constants $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \ldots, b_m$ ...

$$a_n \frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1}u}{dt^{m-1}} + \ldots + b_1 \frac{du}{dt} + b_0 u$$

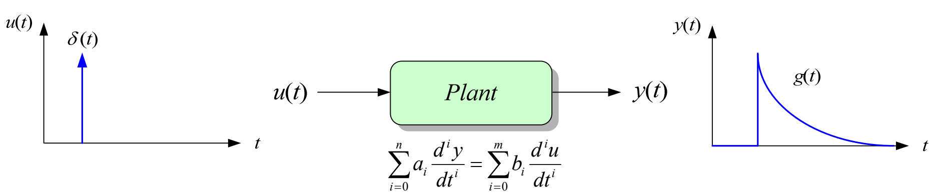
...is termed a **constant coefficient differential equation**

- If none of the coefficients depend explicitly on time, the equation is said to be **time invariant**. This type of equation can be used to describe the dynamic behaviour of a linear time-invariant (LTI) system

- A system is **causal** if its present output depends only on past and present values of its input

Convolution

- The **impulse response** of an LTI system is it's output when subjected to an impulse function, $\delta(t)$



- If the impulse response $g(t)$ of a system is known, it's output $y(t)$ arising from any input $u(t)$ can be computed using a **convolution integral**

$$y(t) = \int_0^\infty u(\tau)\, g(t-\tau)\, d\tau$$

The Laplace Transform

- The **Laplace transform** is a convenient method for solving differential equations

If $f(t)$ is a real function of time defined for all $t > 0$, the Laplace transform is defined as...

$$F(s) = \mathscr{L}\{f(t)\} = \int_{0^+}^{\infty} f(t)\, e^{-st}\, dt$$

...where s is an arbitrary complex variable

- Convolution $$\mathscr{L}\left\{\int_0^t f_1(\tau)\, f_2(t-\tau)\, d\tau\right\} = F_1(s)\, F_2(s)$$
- Linearity $$\mathscr{L}\{k_1 f_1(t) \pm k_2 f_2(t)\} = k_1 F_1(s) \pm k_2 F_2(s)$$
- Final value theorem $$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\, F(s)$$
- Shifting theorem $$\mathscr{L}\{f(t-T)\} = e^{-sT} F(s)$$

Poles & Zeros

$$a_n \frac{d^n y}{dt^n} + \ldots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m u}{dt^m} + \ldots + b_1 \frac{du}{dt} + b_0 u$$

For zero initial conditions, the CCDE can be written in Laplace form...

$$a_n s^n Y(s) + \ldots + a_1 s Y(s) + a_0 Y(s) = b_m s^m U(s) + \ldots + b_1 s U(s) + b_0 U(s)$$

$$\left(a_n s^n + \ldots + a_1 s + a_0\right) Y(s) = \left(b_m s^m + \ldots + b_1 s + b_0\right) U(s)$$

The dynamic behaviour of the system is characterised by the two polynomials

$$B(s) = b_m s^m + b_{m-1} s^{m-1} + b_{m-2} s^{m-2} + \ldots + b_1 s + b_0$$

$$A(s) = a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \ldots + a_1 s + a_0$$

- The roots of $B(s)$ are called the **zeros** of the system
- The roots of $A(s)$ are called the **poles** of the system

The Transfer Function

$$U(s) \longrightarrow \boxed{G(s)} \longrightarrow Y(s)$$

The ratio $\frac{B(s)}{A(s)}$ is called the **transfer function** of the system

$$G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + b_{m-2} s^{m-2} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \ldots + a_1 s + a_0}$$

Numerator & denominator can be factorised to express the transfer function in terms of poles & zeros

$$G(s) = k \frac{(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)}$$

- The quantity $n - m$ is called the **pole excess** of the system
- For a system to be **physically realizable** it must satisfy the constraint $m \leq n$

Tutorial 1.1

Classical First & Second Order Systems

$$Y(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0} U(s)$$

The n^{th} order transfer function gives rise to n roots through partial fraction expansion.

$$Y(s) = \frac{\varepsilon_1}{s+p_1} + \frac{\varepsilon_2}{s+p_2} + \ldots + \frac{\varepsilon_n}{s+p_n}$$

Roots may be either real or appear as complex conjugate pairs. Hence both first and second order terms can be present. This motivates the study of “classical” systems of first and second order.

$$y(t) = \varepsilon_1 e^{-p_1 t} + \varepsilon_2 e^{-p_2 t} + \ldots + \varepsilon_n e^{-p_n t}$$

The time response will be dominated by those roots with the smallest absolute real part.

First Order Systems

The dynamics of a classical first order system are given by the differential equation

$$\tau\,\dot{y}(t) + y(t) = u(t)$$

...where τ is the **time constant** of the system

Taking Laplace transforms and re-arranging to find the transfer function

$$\tau\, s\, Y(s) + Y(s) = U(s)$$

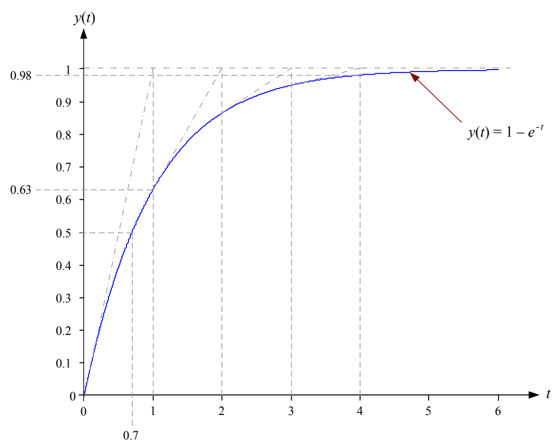
$$\frac{Y(s)}{U(s)} = \frac{1}{s\tau + 1}$$

The output $y(t)$ for any input $u(t)$ can be found using the method of Laplace transforms

$$y(t) = \mathscr{L}^{-1}\left\{U(s)\frac{1}{s\tau + 1}\right\}$$

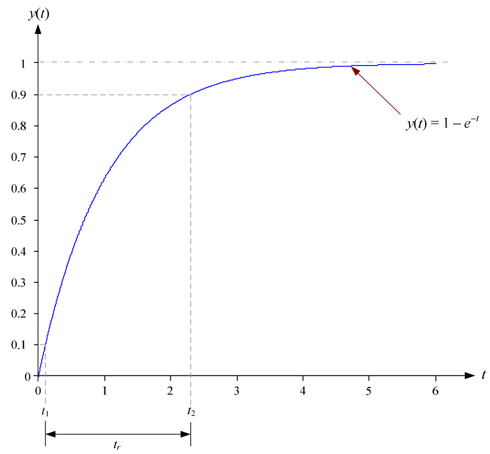
The response following a unit step input is $y(t) = 1 - e^{-\frac{t}{\tau}}$

First Order Unit Step Response



- The output reaches 50% of the final value after ~0.7 τ seconds
- The output reaches ~63% of the final value after τ seconds
- The output $y(t)$ is within 2% of the final value for $t > 4\tau$
- A tangent to the $y(t)$ curve at time $t = 0$ always meets the final value line τ seconds later

Time Constant



- The 10% to 90% **rise time** is given by $t_r = t_2 - t_1$
- The time taken to reach a given output is $t = -\tau \ln(1 - y(t))$
- The rise time for any first order system is approximately $2.2\,\tau$

Second Order Systems

- Linear constant coefficient second-order differential equations of the form

$$\frac{d^2y(t)}{dt^2} + 2\zeta\omega_n \frac{dy(t)}{dt} + \omega_n^2 y(t) = \omega_n^2 u(t)$$

are important because they can often be used to approximate high order systems

- Performance is defined by two quantities:

ζ is called the **damping ratio**

ω_n is called the **un-damped natural frequency**

- The Laplace transform of this second order system is

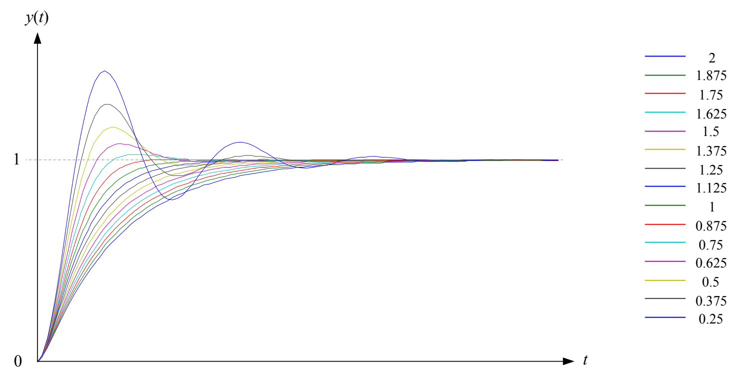
$$\frac{Y(s)}{U(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

...from which we get the characteristic equation...

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

Damping Ratio

- The second order step response can exhibit simple exponential decay, or over-shoot and oscillation, depending on value of the damping ratio ζ
- The ability to represent both types of response means higher order systems can sometimes be approximated using a second order model based on the dominant poles



Unit step response of second order system with varying damping ratio

Second Order Systems

- The poles of the second-order linear system are the solutions of the characteristic equation

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

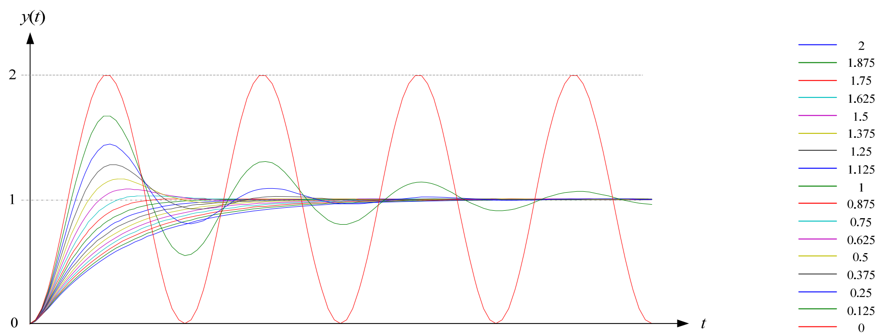
$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

- Four cases of practical interest can be identified, according to the value of ζ

1. ($\zeta > 1$) two real, un-equal negative roots at $s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$
2. ($\zeta = 1$) two real, equal negative roots at $s = -\omega_n$
3. ($0 < \zeta < 1$) a pair of complex conjugate roots at $s = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}$
4. ($\zeta = 0$) two imaginary roots at $s = \pm j\omega_n$

Classification of Second Order Systems

	Damping ratio	Roots	Classification
1	$\zeta > 1$	$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$	*over-damped*
2	$\zeta = 1$	$s = -\omega_n$	*critically damped*
3	$0 < \zeta < 1$	$s = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}$	*under-damped*
4	$\zeta = 0$	$s = \pm j\omega_n$	*un-damped*



The Under-Damped Response

- In the under-damped case ($0 < \zeta < 1$) we have two complex conjugate roots at

$$s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

- Real and imaginary parts are denoted $s = -\sigma \pm j\omega_d$

  τ is the **time constant** of the system: $\tau = \dfrac{1}{\sigma}$

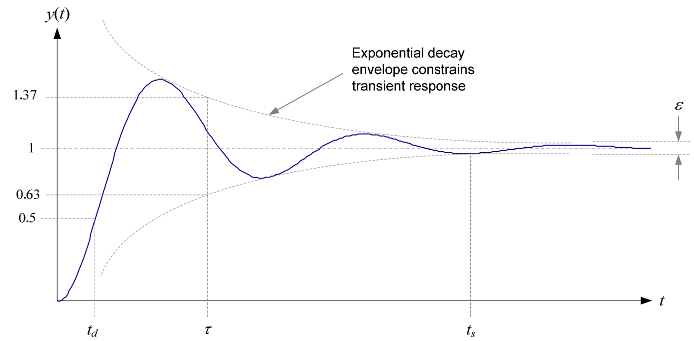
  ω_d is the **damped natural frequency** of the system: $\omega_d = \omega_n\sqrt{1-\zeta^2}$

- The under-damped step response will be of the form

$$y(t) = 1 - \frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \phi) \qquad \text{...where } \phi = \cos^{-1}\zeta$$

Transient Response Decay Envelope

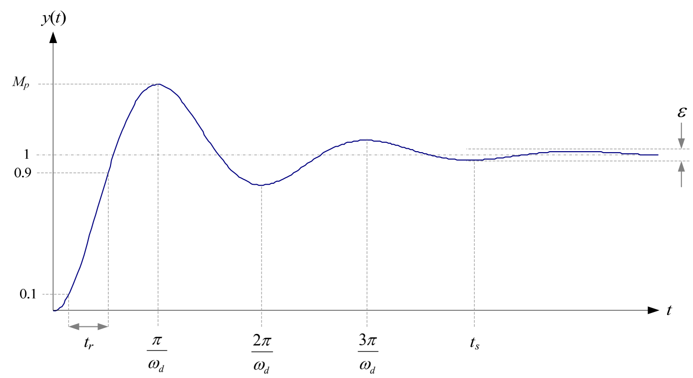
The unit step response of an under-damped second order system is shown below. The transient part consists of a phase shifted sinusoid constrained within an exponential decay envelope.



- Decay rate is determined by the time constant of the system (τ)
- Settling time is determined by the time taken by the envelope to decay to a given error band (ε)

Under-Damped Unit Step Response

The unit step response for the second order case with $0 < \zeta < 1$ is $y(t) = 1 - \dfrac{e^{-\sigma t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \phi)$



- An approximation for rise time is $t_r \approx \dfrac{0.8 + 2.5\zeta}{\omega_n}$
- Settling time is given by $t_s = -\dfrac{\ln\left(\varepsilon\sqrt{1-\zeta^2}\right)}{\sigma}$

Transient Response Peaks

The unit step response of the classical second order system is given by

$$y(t) = 1 - \frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \phi)$$

To find the peaks and troughs, differentiate with respect to time and equate the result to zero.

$$\dot{y}(t) = \frac{\sigma e^{-\sigma t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \phi) - \frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}} \omega_d \cos(\omega_d t + \phi) = 0$$

$$\zeta \sin(\omega_d t + \phi) = \sqrt{1-\zeta^2} \cos(\omega_d t + \phi)$$

$$\tan(\omega_d t + \phi) = \frac{\sqrt{1-\zeta^2}}{\zeta}$$

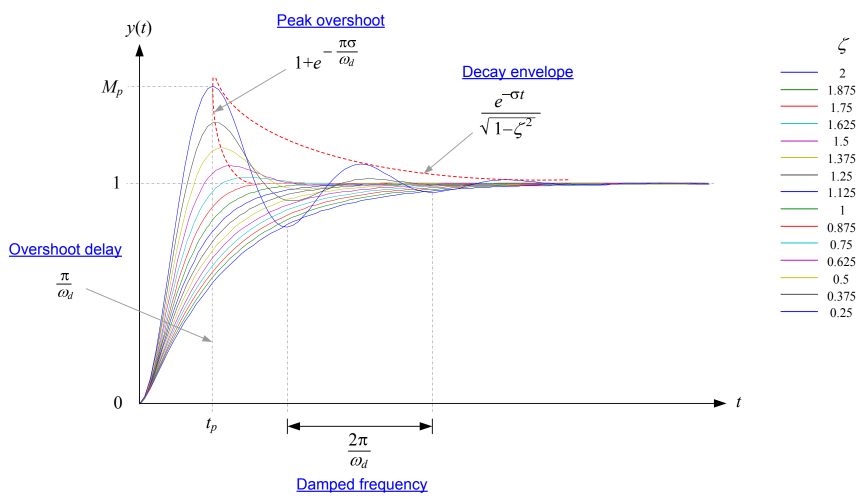
The RHS of this equation is recognisable from $\phi = \tan^{-1}\left(\frac{\omega_d}{\sigma}\right) = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$

$$\tan(\omega_d t + \phi) = \tan(\phi)$$

So the peaks & troughs occur when $\omega_d t = n\pi$ or $t = \frac{n\pi}{\omega_d}$ ...for integer $n > 0$

Second Order Step Response

$$\boxed{y(t) = 1 - \frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \phi)}$$



Second Order Pole Interpretation

For the classical second order system $G(s) = \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

...we have the characteristic equation $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$

For $0 < \zeta < 1$, we have two roots at $s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$

or $s = -\sigma \pm j\omega_d$

These have physical meaning in terms of the unit step response

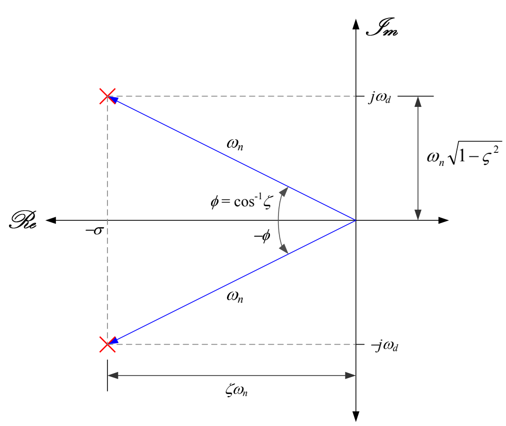
$$y(t) = 1 - \frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \phi)$$

...where the parameters are:

- decay rate: $\sigma = \zeta\omega_n$
- damped natural frequency: $\omega_d = \omega_n\sqrt{1-\zeta^2}$

Second Order Pole Interpretation

- Decay rate and damped natural frequency are the real and imaginary components of the poles
- Un-damped natural frequency and phase delay are the modulus and argument of the poles

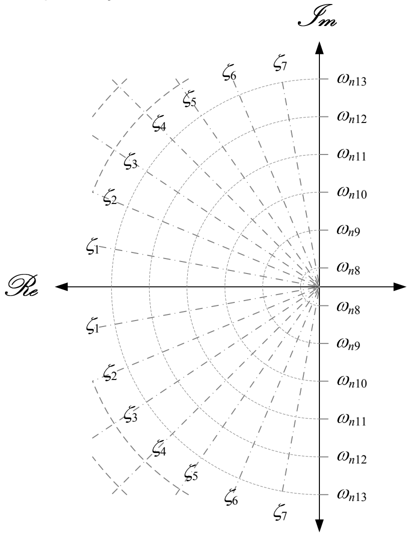


Constant Parameter Loci

- Concentric circles about the origin indicate constant un-damped natural frequency (ω_n)
- Radial lines from the origin indicate constant damping ratio (ζ)

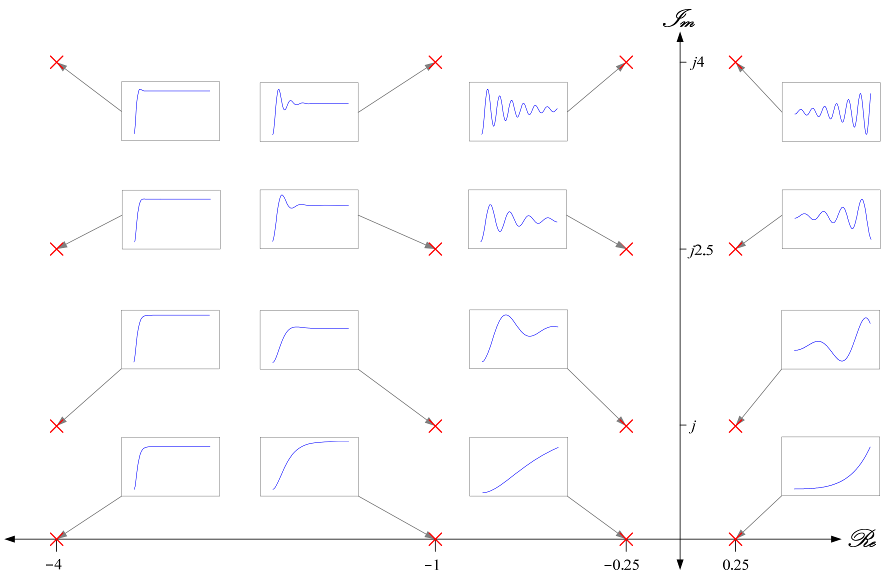
For the second order case, poles are located at

$$s = \omega_n\left(-\zeta \pm j\sqrt{1-\zeta^2}\right)$$



These are the usual grid lines drawn on a pole-zero map to aid in identifying transient response based on dominant pole location.

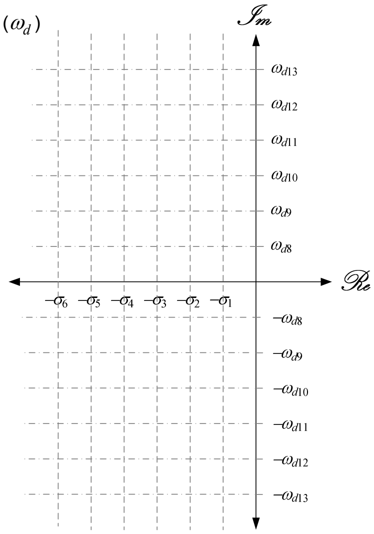
Pole Location vs. Step Response



Upper left quadrant of complex plane shown

Constant Parameter Loci

- Vertical lines indicate constant decay rate (σ)
- Horizontal lines indicate constant damped natural frequency (ω_d)



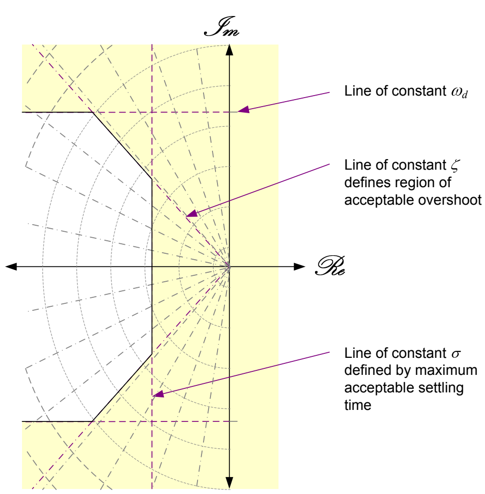
For the second order case, poles are located at

$$s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

$$s = -\sigma \pm j\omega_d$$

- Poles located further to the left have faster decay rate
- Poles with larger imaginary component are more oscillatory

Example Pole Specifications



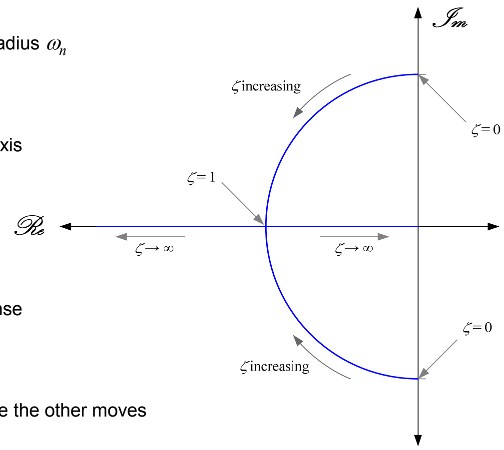
Shaded regions excluded from design because system will not meet transient requirements

Poles positioned far to the left have fast response but require large control effort

Root Locus for Varying Damping

Root locus for the system $G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ with fixed ω_n as ζ varies from 0 to infinity.

- For $\zeta = 0$ both roots are imaginary
- For $0 < \zeta < 1$ the locus is a semi-circle with radius ω_n
- For $\zeta = 1$ the roots are equal and real
- For $\zeta > 1$ both roots lie on the negative real axis



These cases correspond to the transient response classifications given earlier.

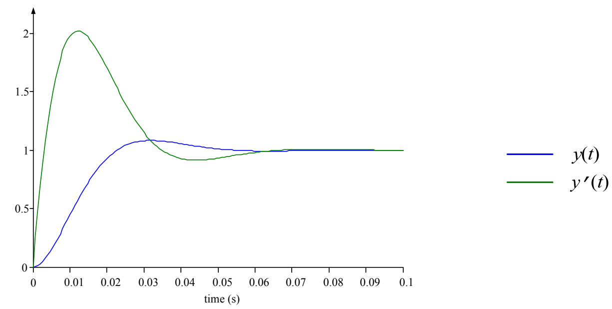
As $\zeta \to \infty$ one root approaches the origin while the other moves to the left along the negative real axis

Effect of LHP Zero

Adding a LHP zero at s = -z to the original transfer function: $G'(s) = \left(\frac{s}{z} + 1\right) G(s)$

For a unit step input, $Y'(s) = \frac{1}{s} G'(s)$ $\qquad Y'(s) = \frac{1}{s} Y(s) + \frac{1}{z} s\, Y(s)$

Taking inverse Laplace transforms $y'(t) = y(t) + \frac{1}{z} \dot{y}(t)$

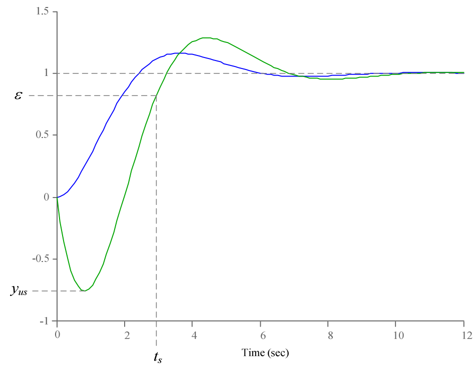


- The effect of adding a LHP zero is to add a scaled version of the derivative of the step response of the original system
- Rise time is decreased and overshoot increases

Effect of RHP Zero

RHP zeros imply inverse response behaviour in the time domain. The step response of a stable plant with n real RHP zeros will cross zero (it's original value) at least n times

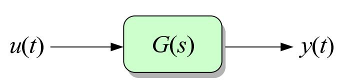
- For a single RHP zero, peak undershoot is bounded by the inequality $|y_{us}| \geq |y_f| \dfrac{1-\varepsilon}{e^{\alpha t_s}-1}$



y_f = final value
t_s = settling time
ε = error bound
α = Re(z)

- The effect of adding an RHP zero is to increase rise time (make the response slower) and induce undershoot

Frequency Response



- If the steady state sinusoid $u(t) = u_0 \sin(\omega t + \alpha)$ is applied to the linear system $G(s)$, then the output is

$$y(t) = y_0 \sin(\omega t + \beta)$$

- The amplitude of output $y(t)$ is modified by $\dfrac{y_0}{u_0}$

- The output $y(t)$ is phase shifted with respect to $u(t)$ by $\phi = \beta - \alpha$

- Amplitude and phase change from input to output are determined by $G(s)$:

$$\boxed{\frac{y_0}{u_0} = |G(j\omega)|} \qquad \boxed{\phi = \angle G(j\omega)}$$

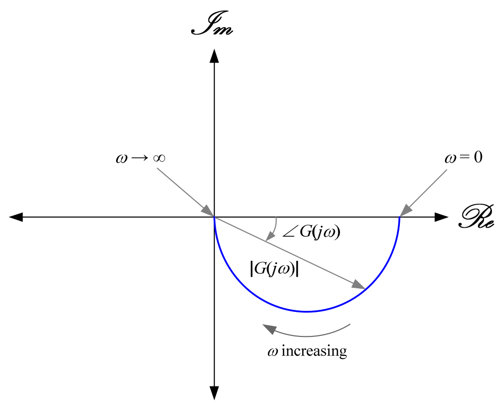
Polar Plots

We can plot the evolution of magnitude and phase of $G(j\omega)$ over frequency as a vector in the complex plane. This is known as a **polar plot**.

For the simple lag system $G(s) = \dfrac{1}{\tau s + 1}$

$$|G(j\omega)| = \frac{1}{\sqrt{1+\omega^2\tau^2}}$$

$$\angle G(j\omega) = -\tan^{-1}(\omega\tau)$$



When $\omega = 0$: $G(j\omega) = 1\angle 0$

As $\omega \to \infty$ $G(j\omega) \to 0\angle -\dfrac{\pi}{2}$

The polar plot for a first order simple lag system stays in the fourth quadrant of the s-plane

Second Order System Polar Plot

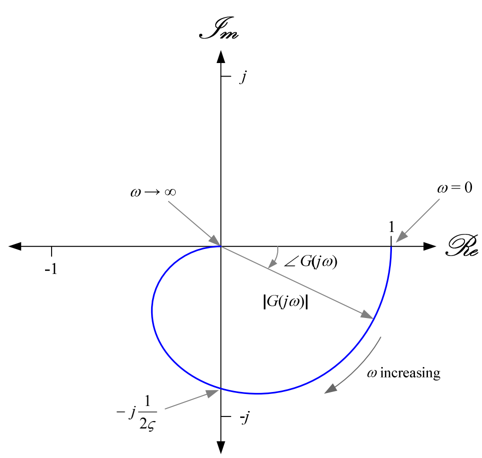
For the classical second order system

$$G(j\omega) = \frac{1}{1 - \dfrac{\omega^2}{\omega_n^2} + j2\zeta\dfrac{\omega}{\omega_n}}$$

Defining $v = \dfrac{\omega}{\omega_n}$ ...

$$|G(jv)| = \frac{1}{\left[(1-v^2)^2 + (2\zeta v)^2\right]^{\frac{1}{2}}}$$

$$\angle G(jv) = -\tan^{-1}\frac{2\zeta v}{1-v^2}$$



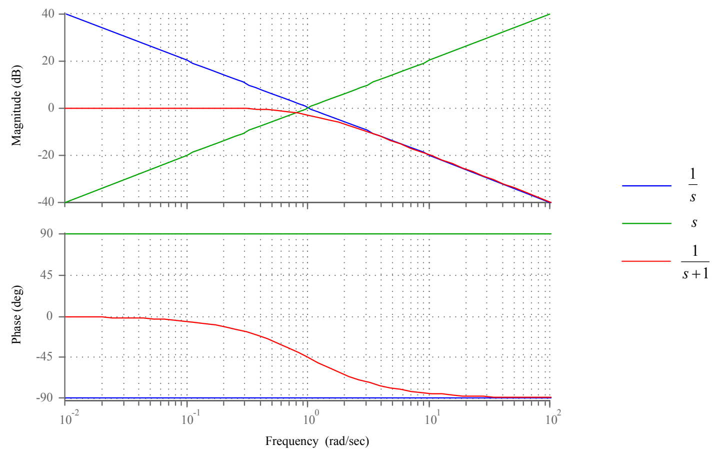
When $\omega = 0$ $G(j\omega) = 1\angle 0$

When $\omega = \omega_n$ $G(j\omega) = \dfrac{1}{2\zeta}\angle -\dfrac{\pi}{2}$

As $\omega \to \infty$ $G(j\omega) \to 0\angle -\pi$

The polar plot of a second order system ($n - m = 2$) enters the third quadrant of the s-plane

The Bode Plot



Bode plots show gain and phase on separate graphs against a common logarithmic frequency axis

For simple transfer functions the gain and phase curves are related by a derivative

Minimum Phase Systems

- A linear system is called **minimum phase** if all its poles and zeros lie in the left half of the s-plane and its response exhibits no time delay

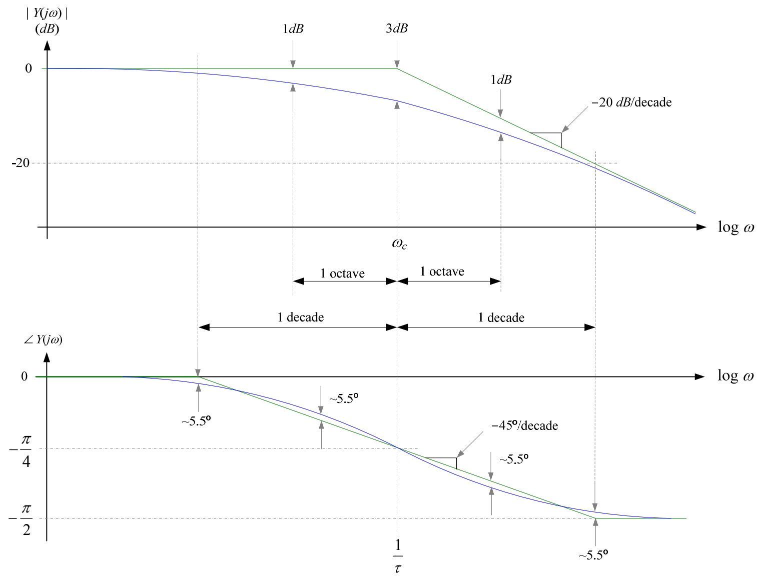
- **Non-minimum phase** systems exhibit more phase lag and are therefore harder to control

- For minimum phase systems the phase curve is related to the rate of change of the gain curve and *vice-versa*.

$$\angle G(j\omega) \approx \frac{\pi}{2} \frac{d \log|G(j\omega)|}{d \log(\omega)}$$

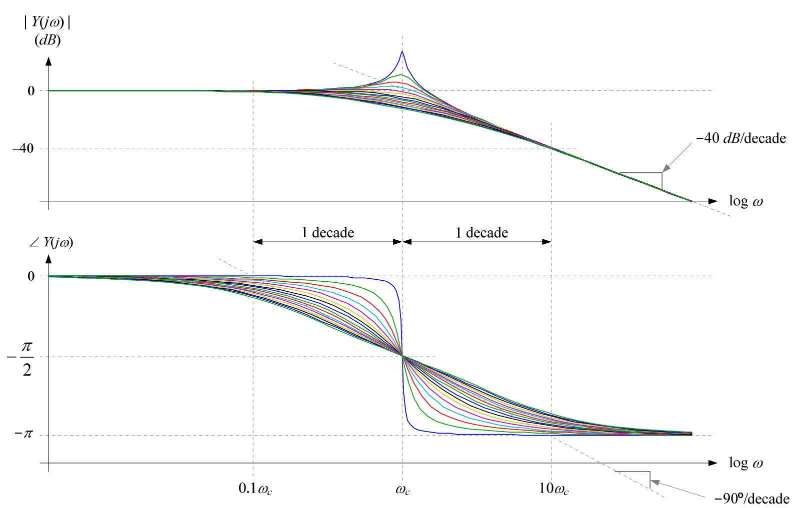
- If the gain curve has a constant slope n, the phase curve has constant value $n\frac{\pi}{2}$

Bode Asymptotes – First Order



Bode asymptotes for the first order case

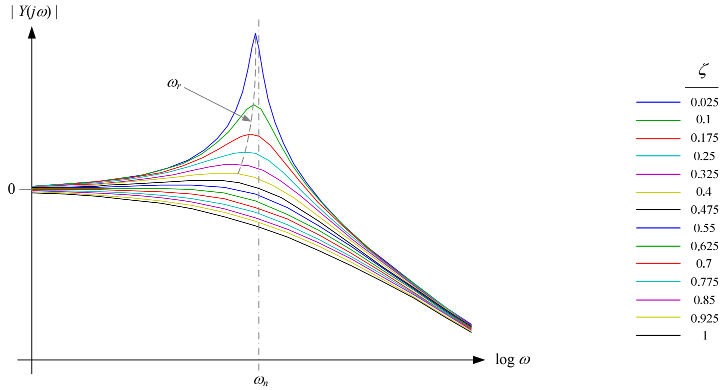
Bode Asymptotes – Second Order



Bode asymptotes for the second order case – plots shown for varying damping ratio

Resonant Peak

- The resonant peak occurs at frequency $\omega_r = \omega_n\sqrt{1-2\zeta^2}$ $\quad : 0 < \zeta < \frac{1}{\sqrt{2}}$
- Resonant peak magnitude is given by $M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$



The resonant peak ω_r approaches ω_n as damping ratio approaches zero: $\omega_r \rightarrow \omega_n$ as $\zeta \rightarrow 0$

Second Order Resonant Peak

For the classical second order system $G(j\omega) = \frac{\omega_n^2}{(\omega_n^2 - \omega^2) + j2\zeta\omega\omega_n}$

Resonant frequency can be found by differentiating with respect to normalised frequency $v = \frac{\omega}{\omega_n}$

The resonant peak occurs at normalised frequency $\frac{d|G(jv)|}{dv} = 0$

...which is at $\omega_r = \omega_n\sqrt{1-2\zeta^2}$

The peak value M_r can be found by substituting the normalised frequency ω_r into the magnitude equation

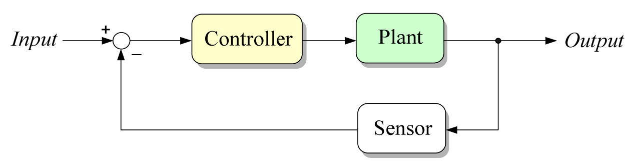
$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \qquad \text{...for } 0 < \zeta < \frac{1}{\sqrt{2}}$$

2. Feedback Control

- **Effects of feedback**
- **Steady state error**
- **Stability & bandwidth**

Effects of Feedback

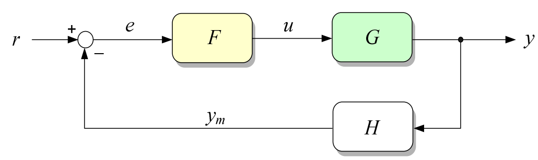
The objective of control is to make the output of a plant precisely follow a reference input



When properly applied, feedback can...

- Reduce or eliminate steady state error
- Reduce the sensitivity of the system to parameter changes
- Change the gain or phase of the system over some desired frequency range
- Cause an unstable system to become stable
- Reduce the effects of load disturbance and noise on system performance
- Linearise a non-linear component

Feedback System Notation



r = reference input

e = error signal

u = control effort

y = output

y_m = feedback

e / r = error ratio

y_m / r = primary feedback ratio

F = ***controller***

G = ***plant***

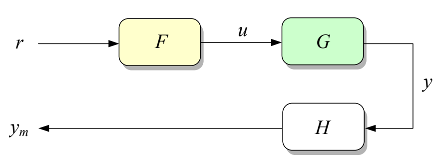
FG = ***forward*** path

H = ***feedback*** path

$L = FGH$ = ***open loop***

y / r = ***closed loop***

The Loop Transfer Function



- The loop transfer function is obtained by breaking the loop at the summing point

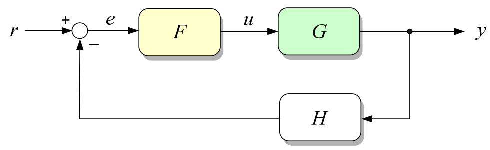
$$L = FGH$$

- If a sinusoid of frequency ω_0 is applied at r, in the steady state the signal y_m will also be a sinusoid of the same frequency ω_0 but it's amplitude and phase will be modified by $L(j\omega_0)$

$$y_m(\omega_0) = L(j\omega_0)\, r(\omega_0)$$

- The condition necessary to sustain oscillation inside the loop is $|L(j\omega_\pi)| \geq -1$ where ω_π is the frequency at which phase lag is π

Negative Feedback



Error equation is: $e = r - Hy$

Output equation is: $y = FGe$

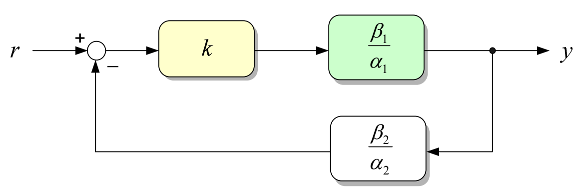
Combining error and output equations gives $y = FG(r - Hy)$

- The open loop transfer function is $L = FGH$
- The closed loop transfer function is $\dfrac{y}{r} = \dfrac{FG}{1 + FGH}$

$$\boxed{\dfrac{\text{direct}}{1 + \text{loop}}}$$

The Closed Loop Transfer Function

Define the transfer functions of the forward and feedback elements as $F = k$, $G = \dfrac{\beta_1}{\alpha_1}$ and $H = \dfrac{\beta_2}{\alpha_2}$



$$\frac{y}{r} = \frac{k\dfrac{\beta_1}{\alpha_1}}{1 + k\dfrac{\beta_1}{\alpha_1}\dfrac{\beta_2}{\alpha_2}}$$

The closed loop transfer function is $$\boxed{\frac{y}{r} = \frac{k\,\beta_1\,\alpha_2}{\alpha_1\,\alpha_2 + k\,\beta_1\,\beta_2}}$$

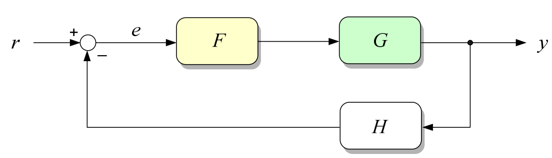
Closed Loop Poles & Zeros

$$\frac{y}{r} = \frac{k\,\beta_1\,\alpha_2}{\alpha_1\,\alpha_2 + k\,\beta_1\,\beta_2}$$

- Closed loop <u>zeros</u> are the roots of $k\,\beta_1\,\alpha_2 = 0$
- Closed loop <u>poles</u> are the roots of the characteristic equation $\alpha_1\,\alpha_2 + k\,\beta_1\,\beta_2 = 0$

- If any $\alpha(s)$ have positive real parts the open loop will be unstable, but the closed loop <u>can</u> be stable
- Closed loop zeros comprise the <u>zeros</u> of the forward path and the <u>poles</u> of the feedback path
- Zeros in the feedback path affect the poles of the closed loop system
- Poles in the feedback path appear as zeros in the closed loop system

Classification by Type



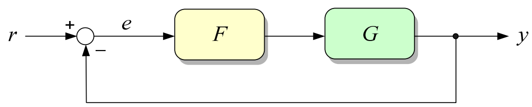
A canonical feedback system with open-loop transfer function

$$FGH = \frac{k_1 \prod_{i=1}^{m}(s+z_i)}{s^l \prod_{i=1}^{n}(s+p_i)} = \frac{k_1\,\beta(s)}{s^l\,\alpha(s)}$$

...where $l \geq 0$ and $-z_i$ and $-p_i$ are non-zero finite zeros and poles of FGH, is called a type l system.

- The type number reflects the number of integrators in the open-loop transfer function
- The steady state error will be either zero, finite or infinite, depending on the integer l

The Steady State Condition



The steady state error following an input $r(s) = \frac{1}{s^n}$ is given by $e_{ss} = \lim_{s \to 0} s \left\{ \frac{1}{s^n} \frac{1}{1 + L(s)} \right\}$

$$e_{ss} = \frac{1}{\lim_{s \to 0} s^{n-1} L(s)}$$

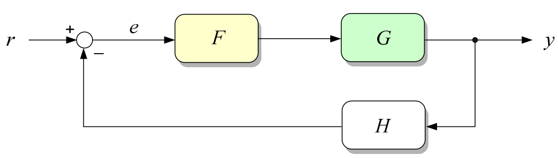
If L represents a type-l system: $L(s) = \frac{k_1}{s^l} \frac{\beta(s)}{\alpha(s)}$

$$\boxed{e_{ss} = \frac{1}{\lim_{s \to 0} s^{n-1-l} k_2}}$$

...where $k_2 = k_1 \frac{\beta(0)}{\alpha(0)}$

- Steady state error is zero if L contains at least $n - 1$ integrators: $e_{ss} = \begin{cases} \infty \text{ if } l < n-1 \\ k_2 \text{ if } l = n-1 \\ 0 \text{ if } l > n-1 \end{cases}$

Error Ratio



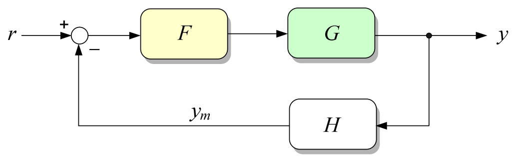
Two ratios play an important role in closed loop performance:

The ***error ratio*** is $\frac{e}{r} = \frac{1}{1 + FGH} = \frac{1}{1 + L}$

- The error ratio determines loop sensitivity to disturbance and is called the ***sensitivity function***

$$\boxed{S = \frac{1}{1 + L}}$$

Feedback Ratio



The ***feedback ratio*** is $$\frac{y_m}{r} = \frac{FGH}{1+FGH} = \frac{L}{1+L}$$

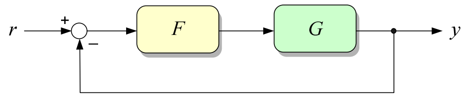
- The feedback ratio determines the reference tracking accuracy of the loop and is called the ***complementary sensitivity function***.

$$\boxed{T = \frac{L}{1+L}}$$

- Closed loop transfer function is related to T by: $\frac{y}{r} = \frac{T}{H}$

Plant Model Sensitivity

For the unity feedback system, the sensitivity function can viewed in a different way.



If we differentiate T with respect to the plant, G...

$$\frac{dT}{dG} = \frac{F(1+FG) - FGF}{(1+FG)^2}$$

$$\frac{dT}{dG} = \frac{F}{(1+FG)^2} = \frac{S\,T}{G}$$

$$\boxed{\frac{dT/T}{dG/G} = S}$$

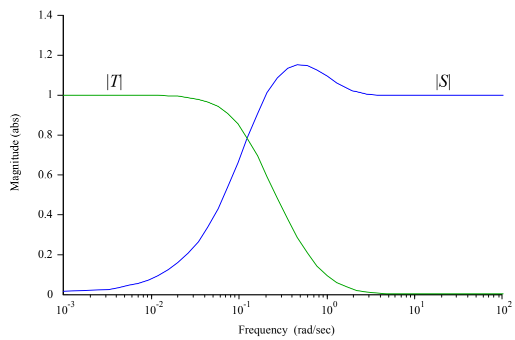
- S is the relative sensitivity of the closed loop to relative plant model error.

S + T = 1

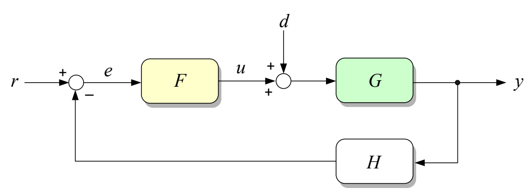
Sensitivity function is: $S = \dfrac{1}{1+L}$ Complementary sensitivity function is: $T = \dfrac{L}{1+L}$

$$\boxed{S + T = \frac{1+L}{1+L} = 1}$$

- The shape of $L(j\omega)$ means we cannot maintain a desired S or T over the entire frequency range



Feedback with Control Disturbance



Superposition allows the effects of output disturbance to be included in y...

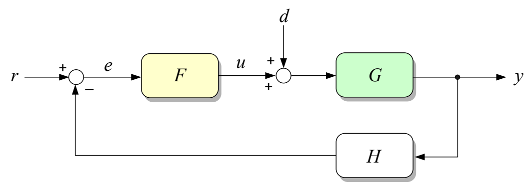
$$y = G(d + F(r - Hy))$$

$$y(1 + FGH) = Gd + FGr$$

Substituting S and T gives

$$\boxed{y = GS\,d + \frac{T}{H}r}$$

Closed Loop Performance



$$y = GS\,d + \frac{T}{H} r$$

For good disturbance rejection we want $S = 0$

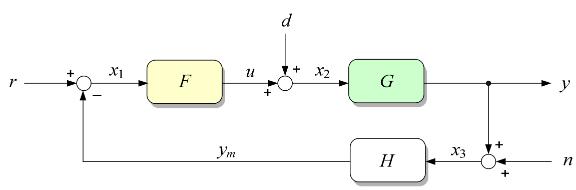
For good reference tracking we want $T = 1$

$$\Rightarrow |L(j\omega)| = \infty \quad \forall \omega$$

If loop gain can be kept very high, then $y \approx \dfrac{r}{H}$

- As $|L(j\omega)|$ rolls off both tracking performance and disturbance rejection will deteriorate
- Controller design involves shaping the open loop plot $L(j\omega)$ to achieve the best compromise between reference tracking and disturbance rejection

Basic Feedback Equations



$$y = G\,(\,d + u\,) \qquad x_1 = r - H\,x_3$$

$$y_m = H\,(\,n + y\,) \qquad x_2 = d + F\,x_1$$

$$u = F\,(\,r - y_m\,) \qquad x_3 = n + G\,x_2$$

The equations for the exogenous signals in terms of internal signals are:

$$r = x_1 + H\,x_3$$

$$d = x_2 - F\,x_1$$

$$n = x_3 - G\,x_2$$

Basic Feedback Equations

The loop equations can be written in matrix form as:

$$\begin{bmatrix} r \\ d \\ n \end{bmatrix} = \begin{bmatrix} 1 & 0 & H \\ -F & 1 & 0 \\ 0 & -G & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

If the $3x3$ matrix is non-singular, the loop equation can be re-arranged to find the three internal signals

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{1+FGH} \begin{bmatrix} 1 & -GH & -H \\ F & 1 & -FH \\ FG & G & 1 \end{bmatrix} \begin{bmatrix} r \\ d \\ n \end{bmatrix}$$

$$x_1 = \frac{1}{1+L} r - \frac{GH}{1+L} d - \frac{H}{1+L} n$$

$$x_2 = \frac{F}{1+L} r + \frac{1}{1+L} d - \frac{FH}{1+L} n$$

$$x_3 = \frac{FG}{1+L} r + \frac{G}{1+L} d + \frac{1}{1+L} n$$

Internal Stability

- **Internal stability** requires that x_1, x_2 & x_3 do not grow without bound, so the following seven transfer functions must all be stable:

$$\frac{1}{1+L} \quad \frac{F}{1+L} \quad \frac{G}{1+L} \quad \frac{H}{1+L}$$

$$\frac{GH}{1+L} \quad \frac{FH}{1+L} \quad \frac{FG}{1+L}$$

These can be written in terms of S and T:

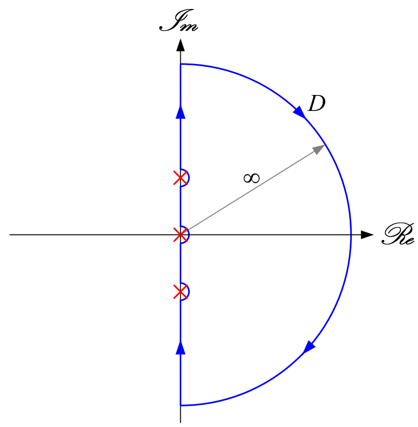
$$S \quad FS \quad GS \quad HS$$

$$\frac{T}{F} \quad \frac{T}{G} \quad \frac{T}{H}$$

- <u>External</u> stability is a pre-requisite for internal stability

The Nyquist D Contour

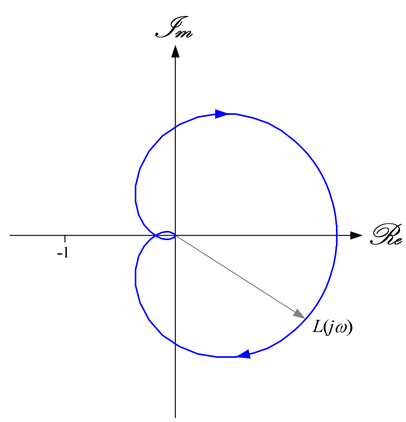
- The **Nyquist D contour** encloses the RHP with small indentations to avoid any poles of $L(s)$ on the imaginary axis and an arc at infinity



- The Nyquist D contour encloses the RHP with small indentations to avoid any poles of $L(s)$ on the imaginary axis and an arc at infinity

The Nyquist Plot

- The **Nyquist plot** is the image of the loop transfer function $L(s)$ when s traverses D in the clockwise direction



- If $L(s)\to 0$ as $s\to\infty$ then the portion of the D contour at infinity maps to the origin

Closed Loop Stability

Closed loop stability can be defined in terms of either pole location or frequency response.

1) Pole location

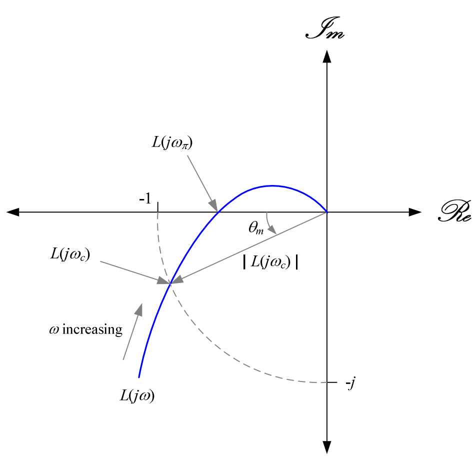
- The poles of the characteristic equation are determined by solving $1 + L(s) = 0$
- For stability, all the system poles must lie in the open left half plane
- The root locus diagram provides an effective means of determining optimum loop gain
- Time delays must first be approximated by rational transfer functions

2) Frequency response

- Plot the gain and phase of the open loop $L(j\omega)$ vs. frequency
- Evaluate gain and phase margins graphically to determine relative stability
- Information clearly presented on Bode or Nyquist plots
- Can represent non-minimum phase systems (but Bode relations do not apply)

Phase Margin

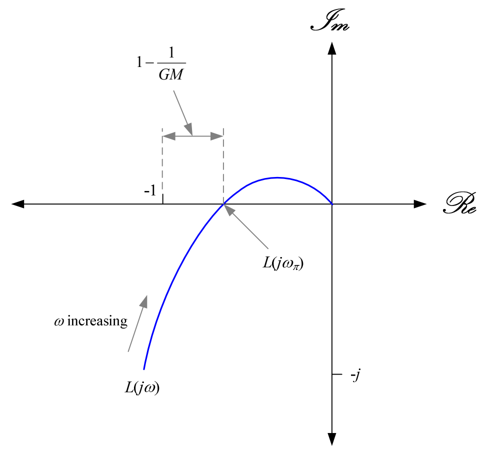
- Phase Margin (PM) is defined as: $PM = \angle L(j\omega_c) + \pi$... where ω_c is the **gain crossover frequency**



Phase margin is the angular difference between the point on the frequency response at the unit circle crossing and -180^o. For a robust control system we want large positive θ_m.

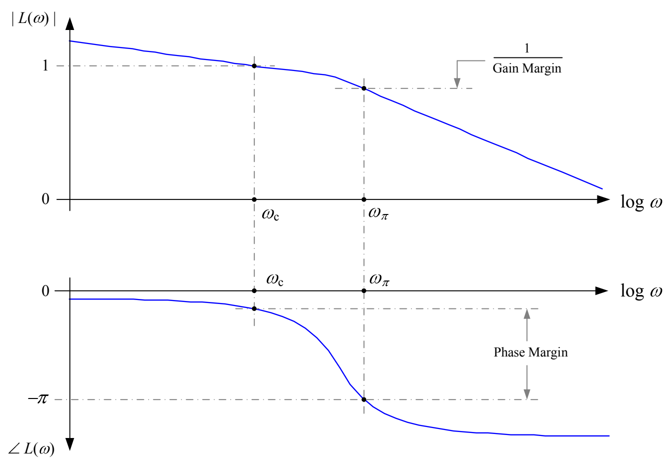
Gain Margin

- Gain Margin (GM) is defined as: $GM = \dfrac{1}{|L(j\omega_\pi)|}$ ... where ω_π is the **phase crossover frequency**



Gain margin is the amount of extra loop gain we would have to add for the $L(j\omega)$ line to reach the $[-1,0]$ point. For a robust control system we want large positive GM.

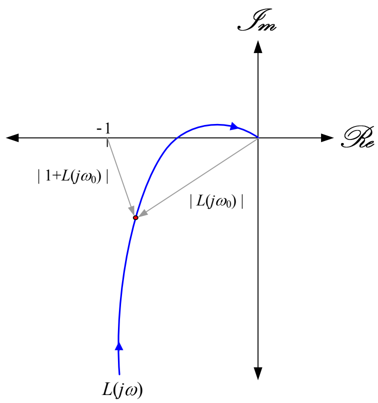
Gain & Phase Margins on Bode Plot



- Gain Margin (GM) is defined as: $GM = \dfrac{1}{|L(j\omega_\pi)|}$ ... where ω_π is the **phase crossover frequency**
- Phase Margin (PM) is defined as: $PM = \angle L(j\omega_c) + \pi$ ... where ω_c is the **gain crossover frequency**

Loop Sensitivity from the Nyquist Plot

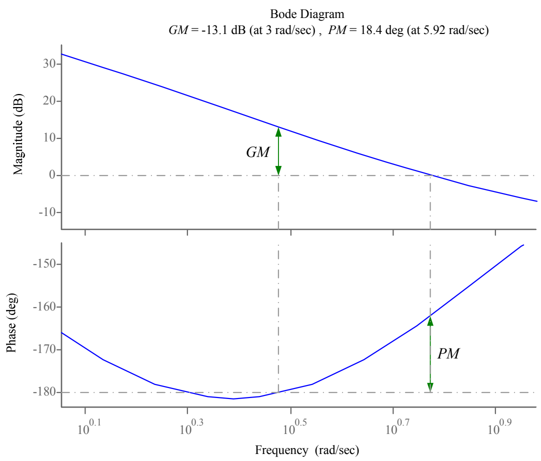
- The vectors $|L(j\omega)|$ and $|1+L(j\omega)|$ can readily be obtained from the Nyquist plot for any frequency ω_0



- Estimation of S & T curves is straightforward from the Nyquist diagram

Conditional Stability

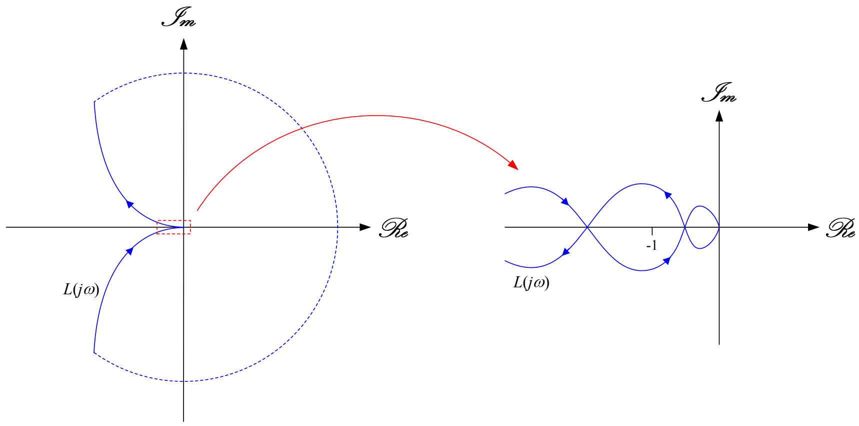
Bode diagram for the system with open loop transfer function $L(s) = \dfrac{3(s+6)^2}{s(s+1)^2}$



- Open loop phase lag exceeds 180 degrees at frequencies below cross-over but closed loop is stable

Conditional Stability

Nyquist diagram for the system with open loop transfer function $L(s) = \dfrac{3(s+6)^2}{s(s+1)^2}$

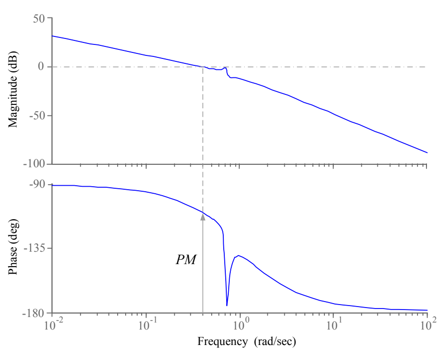


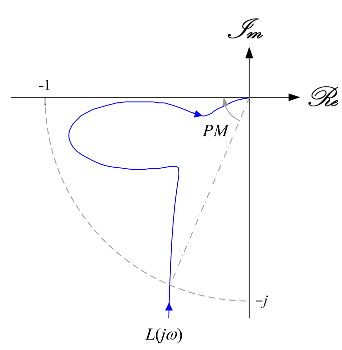
- The plot of $L(s)$ as s traverses the Nyquist contour does not make a clockwise encirclement of the critical point, hence the closed loop system will be stable
- In this case, reducing the loop gain will cause instability

Stability Margins - Example

$$L(s) = \frac{0.38(s^2 + 0.1s + 0.55)}{s(s+1)(s^2 + 0.06s + 0.5)}$$

Although gain and phase margins are adequate (GM infinite, PM = 70 deg.), the system has two lightly damped modes (ζ_1 = 0.81, ζ_2 = 0.014) causing a highly oscillatory step response.



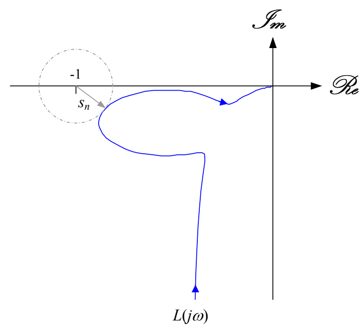


- In general, the Nyquist plot provides more complete stability information than the Bode plot

Stability Index

- A better measure of relative stability is to gauge the closest $L(j\omega)$ passes to the critical point [-1,0]. This is called the **stability index**

$$s_n = \min_{\omega} |1 + L(j\omega)|$$



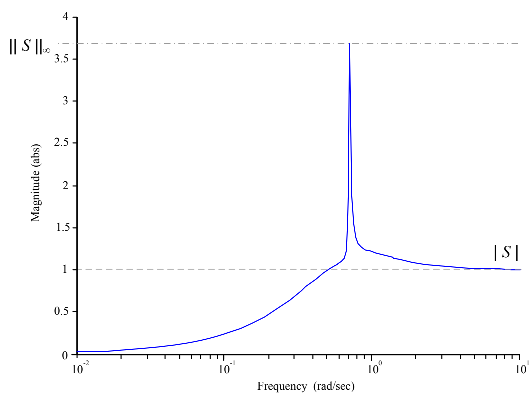
- The reciprocal of the sensitivity index s_n is the infinity norm of the sensitivity function

$$\frac{1}{s_n} = \frac{1}{\min_{\omega} |1 + L(j\omega)|} = \max_{\omega} \left| \frac{1}{1 + L(j\omega)} \right| = \| S \|_\infty$$

- A large value of $\| S \|_\infty$ indicates $L(j\omega)$ comes close to the critical point and the feedback system is nearly unstable

Sensitivity Norm - Example

- The infinity norm of the sensitivity function shows considerable peaking, $| S | >> 1$, indicating that $L(j\omega)$ passes close to the critical point and possible robustness issues

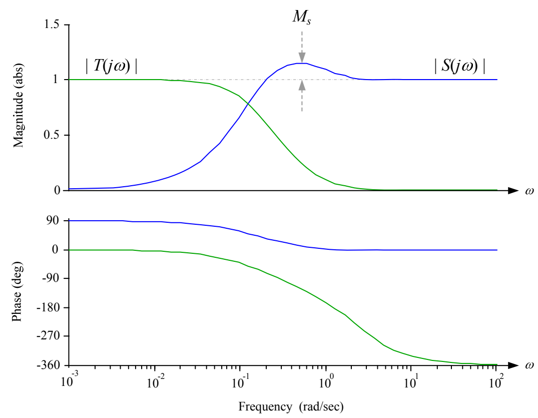


- Although the S infinity norm is a good measure of relative stability and robustness, is does not indicate stability in the binary sense: for example, a Nyquist plot which encloses the critical point might have a similar S response to that shown above

Maximum Peak Criteria

- The maximum peaks of sensitivity and complementary sensitivity are

$$M_S = \max_{\omega} |S(j\omega)| \qquad\qquad M_T = \max_{\omega} |T(j\omega)|$$



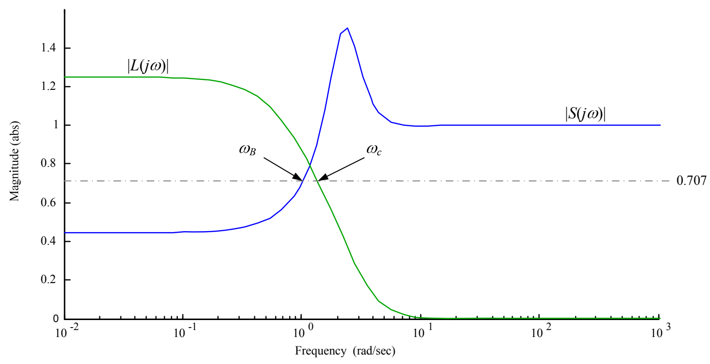
Typical design requirements are: $M_S < 2\ (6dB)$ and $M_T < 1.25\ (2dB)$

Bandwidth

Bandwidth is defined as the frequency range $[\omega_1\ \omega_2]$ over which control is effective: $\omega_B = \omega_1 - \omega_2$

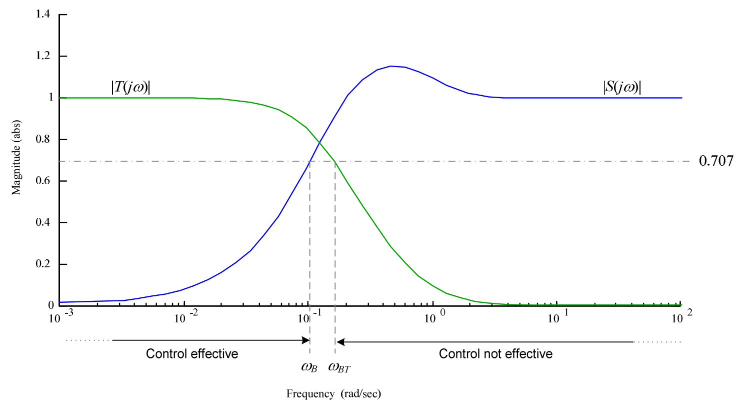
Normally we require steady state performance, so $\omega_1 = 0$ and $\omega_B = \omega_2$

- ***Closed loop bandwidth*** ω_B is defined as the frequency at which $|S(j\omega)|$ first crosses $-3dB$ from below
- ***Gain crossover frequency*** ω_c is defined as the frequency at which $|L(j\omega)|$ first crosses $0dB$ from above



Regions of Control Effectiveness

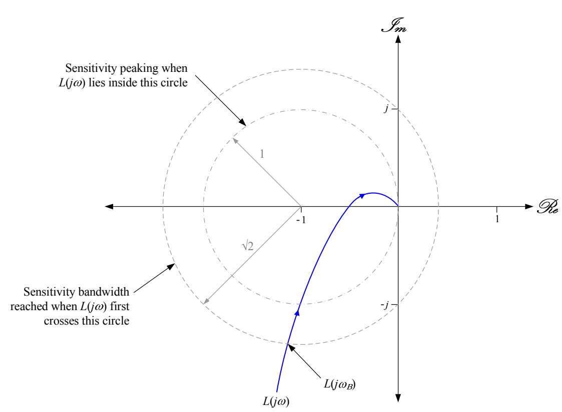
Bandwidth ω_B can be defined in terms of the $|S(j\omega)|$ crossing of $-3dB$ from below.



- Below ω_B performance is improved by control
- Between ω_B and ω_{BT} control affects response but does not improve performance
- Above ω_{BT} control has no significant effect

Nyquist Diagram: Sensitivity Function

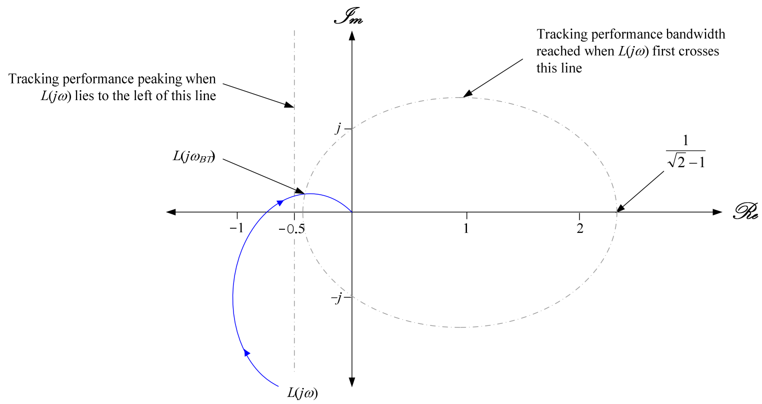
- Peaking & bandwidth properties of the sensitivity function can be inferred from the Nyquist diagram



- Sensitivity bandwidth reached when $|S| = \dfrac{1}{|1+L|}$ first crosses $\dfrac{1}{\sqrt{2}}$ from below: $|S| > \dfrac{1}{\sqrt{2}} \Rightarrow |1+L| < \sqrt{2}$
- Sensitivity peaking occurs when $|1+L| < 1$

Nyquist Diagram: Tracking Performance

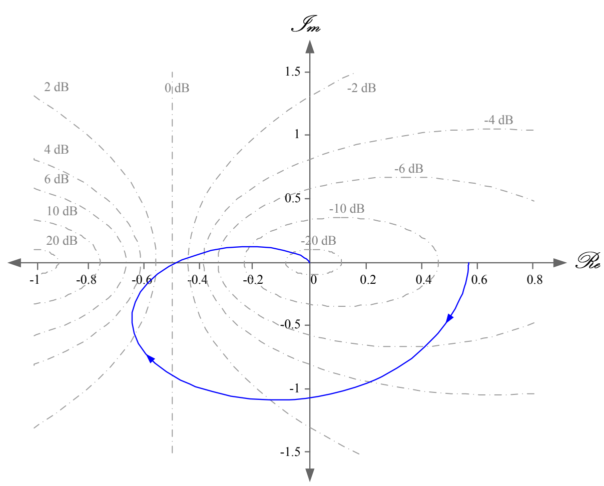
- Peaking & bandwidth of the tracking response can be inferred from the Nyquist diagram



- Tracking bandwidth reached when $|T| = \frac{|L|}{|1+L|}$ first crosses $\frac{1}{\sqrt{2}}$ from above: $|T| < \frac{1}{\sqrt{2}} \Rightarrow |1+L| > \sqrt{2}|L|$

- Tracking peaking occurs when $Re\{\,L(j\omega)\,\} < -0.5$

Nyquist Diagram Grid Lines



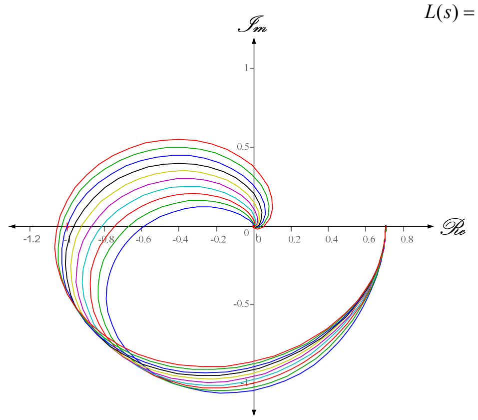
- Conventional Nyquist diagram grid displays peaking of tracking response in dB

- $-3\ dB$ line corresponds to tracking bandwidth locus

Effect of Time Delay

Nyquist plot shows progressively larger time delay added to a stable third order system

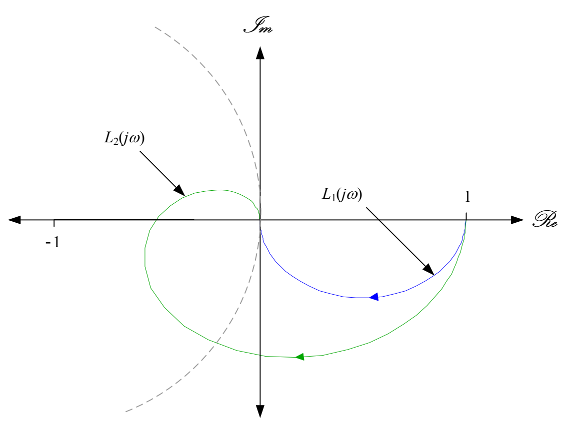
$$L(s) = e^{-\theta s}\frac{6}{(s+2)(s^2+s+4.25)}$$



θ	PM
0	42.1174
0.05	36.1664
0.1	30.2154
0.15	24.2643
0.2	18.3133
0.25	12.3623
0.3	6.4113
0.35	0.460286
0.4	-5.49073
0.45	-11.4417

- Time delay is equivalent to a frequency dependent phase lag which erodes phase margin.

RHP Zero: Additional Phase Lag



The above example shows Nyquist plots for the systems: $L_1(s) = \frac{1}{s+1}$ $L_2(s) = \frac{1}{s+1}\frac{-s+1}{s+1}$

Both systems are strictly proper with pole excess of 1, but $L_2(j\omega)$ has significantly higher phase lag due to its' RHP zero, causing it to enter the unit circle centred at [-1,0].

Loopshaping Limitations

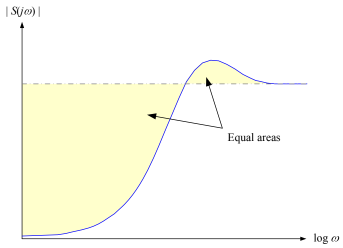
- If $L(s)$ is rational and has a pole excess of at least 2, then for closed-loop stability

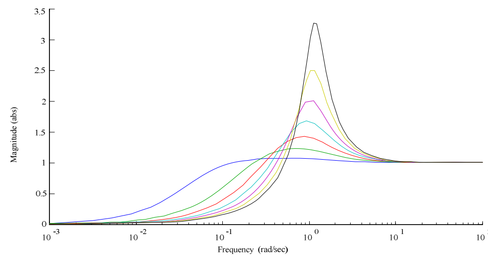
$$\int_0^\infty \ln|S(j\omega)|\, d\omega = \pi \sum_{i=1}^{N} \mathrm{Re}(p_i)$$

...where there are N RHP poles at locations p_i

- For a stable loop... $$\int_0^\infty \ln|S(j\omega)|\, d\omega = 0$$

- For a given system, any increase in bandwidth ($|S|$ < 1 over larger frequency range) must come at the expense of a larger sensitivity peak. This is termed the **waterbed effect**.





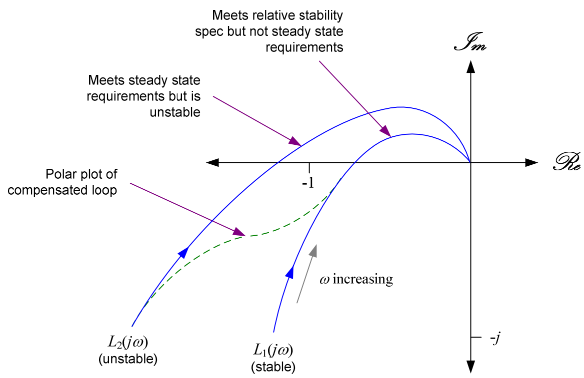
3. Controller Design

- **Phase compensation**
- **Root locus analysis**
- **Transient tuning**

Loop Shaping Requirements

- Requirements for good control are $S \approx 0$ and $T \approx 1$, so $|L|$ must be large
- $|L|$ cannot be maintained large over the entire frequency range so compromises must be made
- Loop shaping means designing the shape of the $L(j\omega)$ gain & phase curves
- Design of $L(j\omega)$ is most critical in the region between ω_c and ω_π:
  - For stability, we need $|L(j\omega_\pi)| < 1$
  - Phase at crossover $\angle L(j\omega_c)$ should be small
  - For fast response, we want large ω_c and ω_π
- For low frequency performance, $L(s)$ must contain at least one integrator for each integrator in r

Phase Compensation

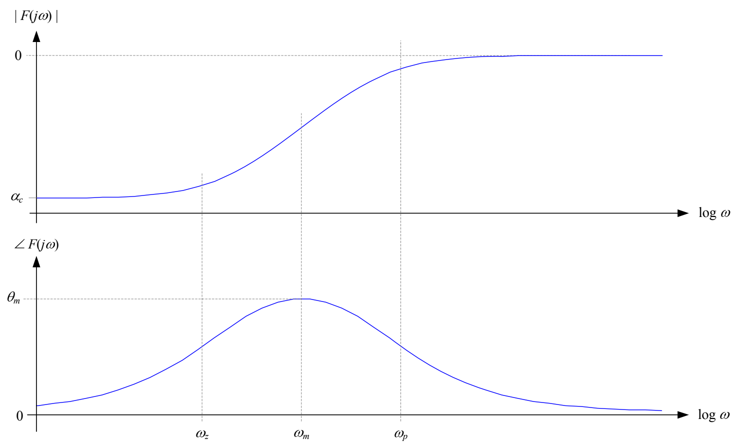


Three types of compensation are commonly used:

- Start with gain k_1 and introduce **phase lead** at high frequencies to achieve specified PM, GM, M_p, ...etc.
- Start with gain k_2 and introduce **phase lag** at low frequencies to meet steady-state requirements
- Start with gain between k_1 and k_2 and introduce phase lag at low frequencies and lead at high frequencies (**lag-lead** compensation)

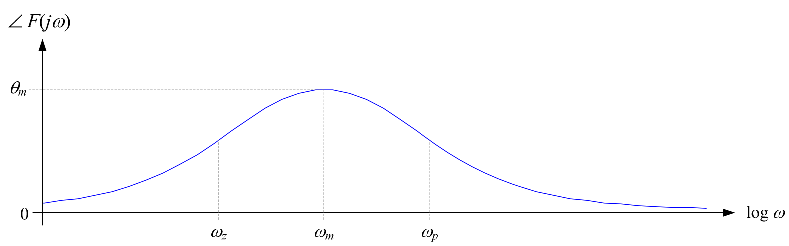
Phase Lead Compensation

- The first order phase lead compensator has one pole and one zero, with the zero frequency lower than the pole
- The simple lead compensator transfer function is $F(s) = \dfrac{s+\omega_z}{s+\omega_p}$ ...where (ω_z < ω_p)



Lead Compensator Design

Define $F(s) = \dfrac{1}{\alpha}\dfrac{1+\alpha cs}{1+cs}$ ...where α > 1



Maximum phase occurs at frequency $\omega_m = \dfrac{1}{\alpha\sqrt{c}}$

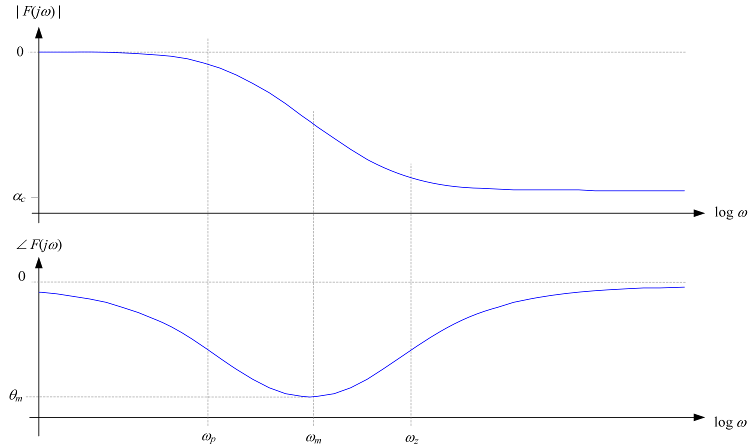
Maximum phase is $\sin\theta_m = \dfrac{\alpha-1}{\alpha+1}$

- If θ_m is known, choose c to fix ω_p, then calculate α using $\alpha = \dfrac{1+\sin\theta_m}{1-\sin\theta_m}$

Tutorials 3.1 & 3.2

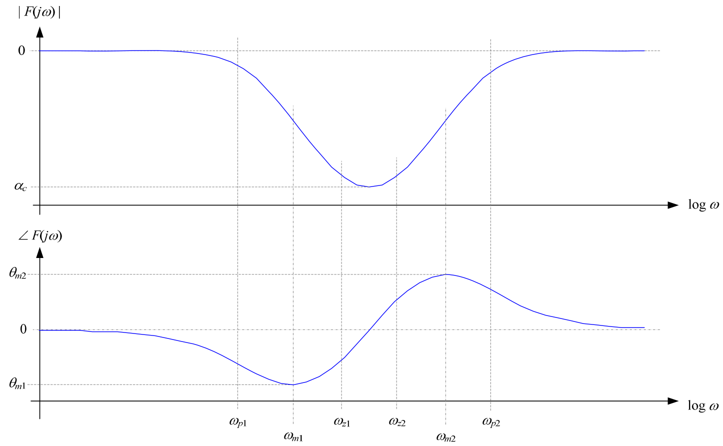
Phase Lag Compensation

- The first order phase lag compensator has one pole and one zero, with the pole frequency lower than the zero
- The simple lag compensator transfer function is $F(s) = \dfrac{s+\omega_z}{s+\omega_p}$ ...where ($\omega_p < \omega_z$)



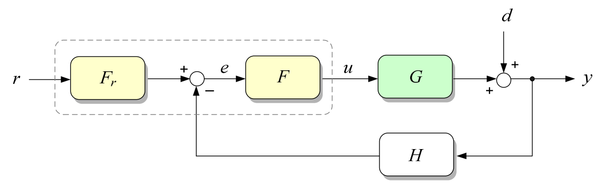
Lag-Lead Compensation

- The lag-lead compensator has two poles and two zeros arranged to induce phase lag and phase lead over different frequency ranges
- The lag-lead compensator transfer function is $F(s) = \dfrac{(s+\omega_{z1})(s+\omega_{z2})}{(s+\omega_{p1})(s+\omega_{p2})}$
- For unity gain: $\omega_{p1}\ \omega_{p2} = \omega_{z1}\ \omega_{z2}$



2DOF Controller

The 2 degrees-of-freedom structure permits disturbance rejection and tracking performance to be designed independently.

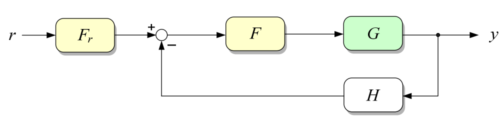


The output equation is $\boxed{y = Sd + F_r \frac{T}{H} r}$

In practice, tuning is a two stage procedure:

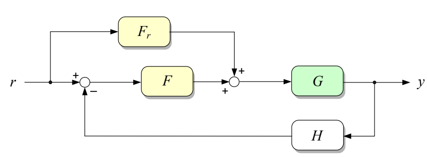
- First, loop shaping is performed to design S for disturbance rejection
- Then the input filter F_r is designed to achieve desired tracking response

Two Degrees of Freedom Controllers



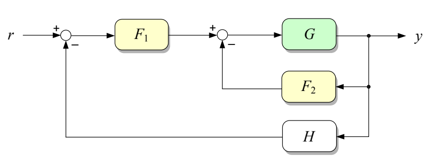
Feed-Forward Control

$$\frac{y}{r} = \frac{F_r F G}{1 + FGH}$$



Feed-Forward Compensation

$$\frac{y}{r} = \frac{G(F_r + F)}{1 + FGH}$$



Parallel Feedback Compensation

$$\frac{y}{r} = \frac{F_1 G}{1 + G(F_2 + F_1 H)}$$

Internal Model Principle

$$r \longrightarrow \boxed{F} \xrightarrow{u} \boxed{G} \longrightarrow y$$

Suppose we have a model of G denoted $\tilde{G}$

The basis of the Internal Model Principle is to determine $\tilde{G}$ and set $F = \tilde{G}^{-1}$

$$r \longrightarrow \boxed{\tilde{G}^{-1}} \xrightarrow{u} \boxed{G} \longrightarrow y$$

Then the output is $y = \left\{ G\,\tilde{G}^{-1} \right\} r$

i.e. if the model if accurate, perfect control is achieved without feedback!

In practice the approach has limited use, since...

- Information about the plant may be inaccurate or incomplete
- The plant model may not be invertible or realisable
- Control is not robust, since any change in the process results in output error

Internal Model Control (IMC)

- An alternative to open loop shaping is to directly synthesize the closed loop transfer function
- The approach is to specify a desirable closed-loop T.F. then find the corresponding controller

$$Q = \frac{FG}{1 + FGH}$$

$$F = G^{-1} \frac{Q}{1 - QH}$$

- This method is known as Internal Model Control, or Q-parameterisation
- In principle, any closed-loop response can be achieved providing knowledge of the plant is accurate and it is invertible
- The plant G might be difficult to invert because...
  - RHP zeros give rise to RHP poles – *i.e.* the controller will be *unstable*
  - time delay becomes time advance – *i.e.* the controller will be *non-causal*
  - if the plant is strictly proper, the inverse controller will be *improper*

IMC – Non-Minimum Phase Plant

- Step 1: factorise G into invertible and non-invertible (*i.e.* non-minimum phase) parts: $G = G_m G_n$

  ...where the non-invertible part G_n is given by $G_n = e^{-\theta s} \prod_{i=1}^{q} \frac{-s+z_i}{s+z_i}$

  This is an all-pass filter with delay. Any new LHP poles in G_n can be cancelled by LHP zeros in G_m

- Step 2: specify the desired closed loop T.F. to include G_n: $Q = f\,G_n$

- Step 3: substitute into the controller equation: $F = G_m^{-1} G_n^{-1} \dfrac{f\,G_n}{1 - f\,G_n H}$

$$\boxed{F = G_m^{-1} \frac{f}{1 - f\,G_n H}}$$

This equation does not require inversion of G_n

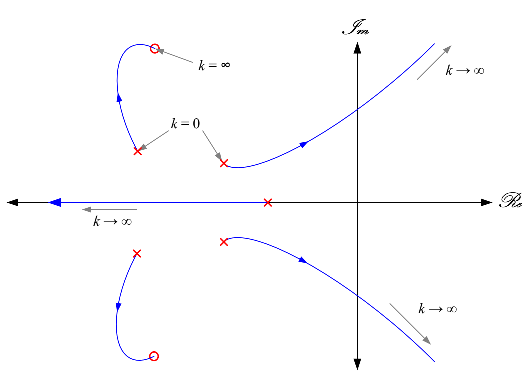
Root Locus Analysis

- A **root locus plot** is a diagram of the <u>closed loop</u> pole paths in the complex plane as loop gain k is varied from 0 to infinity

$$\alpha_1(s)\alpha_2(s) + k\,\beta_1(s)\beta_2(s) = 0$$

- Every locus begins at a pole when $k = 0$, and ends either at a zero or infinity (*i.e.* follows an asymptote)

- The number of asymptotes is given by the pole excess, n - m



Angle & Magnitude Criteria

Points on the root locus are given by $\alpha_1\alpha_2 + k\,\beta_1\beta_2 = 0$

- When $k = 0$, the roots are at $\alpha_1\alpha_2 = 0$ ...*i.e.* at the poles of the open loop system.
- As $k \to \infty$, the roots are at $\beta_1\beta_2 = 0$ ...*i.e.* at the zeros of the open loop system.
- For $0 < k < \infty$, the roots can be found as follows...

$$k\,\beta_1\beta_2 = -\alpha_1\alpha_2$$

$$k\,\frac{\beta_1\beta_2}{\alpha_1\alpha_2} = -1$$

$$FGH = -1$$

i.e. all points on the root locus must satisfy the **magnitude criterion**: $|L| = \frac{1}{k}$

...and the **angle criterion**: $\angle L = -n\pi$ (n odd)

Root Locus Interpretation

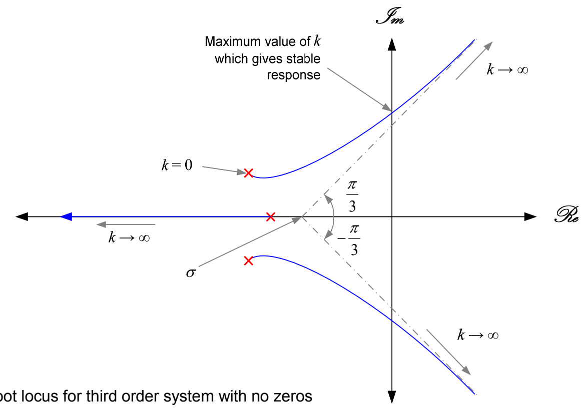
- Complex conjugate poles lead to a step response which is under-damped
- If all closed loop poles are real, the step response will be over-damped
- Closed loop zeros may induce overshoot even in an over-damped system
- The response is dominated by s-plane poles closest to the imaginary axis
- The further left the dominant system poles are...
  - the faster the response
  - the greater the bandwidth
- When a pole and zero nearly cancel each other, the pole has little effect on overall response
- Time domain and frequency domain specifications are loosely related:
  - rise time and bandwidth are inversely proportional
  - larger PM, GM and lower M_p improve damping
- Adding a pole to the forward path pushes the root loci towards the right
- Adding a LHP zero generally moves & bends the root loci towards the left

Tutorial 3.3

Intersection of Loci Asymptotes

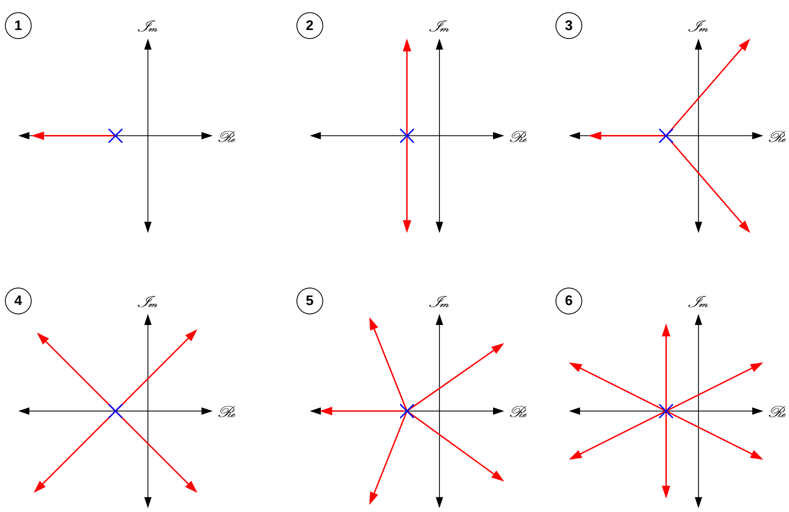
- For large k, the root loci are asymptotes with angles given by $\theta_i = \pi \times \frac{2i+1}{|n-m|}$
- Asymptotes intersect at a point on the real axis given by

$$\sigma = \frac{\Sigma \text{ real parts of the poles of } L(s) - \Sigma \text{ real parts of the zeros of } L(s)}{n - m}$$



Typical root locus for third order system with no zeros

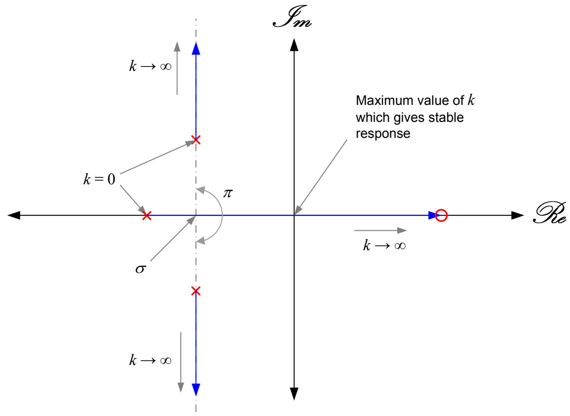
Root Loci Asymptotes by Pole Excess



RHP Zero: High Gain Instability

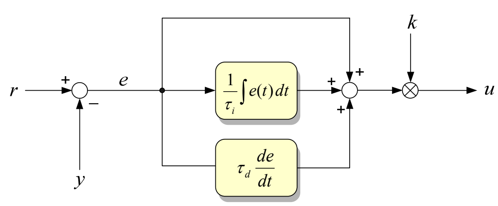
As open loop gain increases, each root locus tend towards either an infinite asymptote or an open loop zero. *i.e.* for proper systems, each zero accommodates a closed loop pole at infinite gain.

- For each RHP zero one locus crosses into the RHP, so at sufficiently high gain the closed loop will become unstable



- Maximum achievable control gain (k_{crit}) occurs where the first closed loop pole crosses into the RHP

PID Controllers



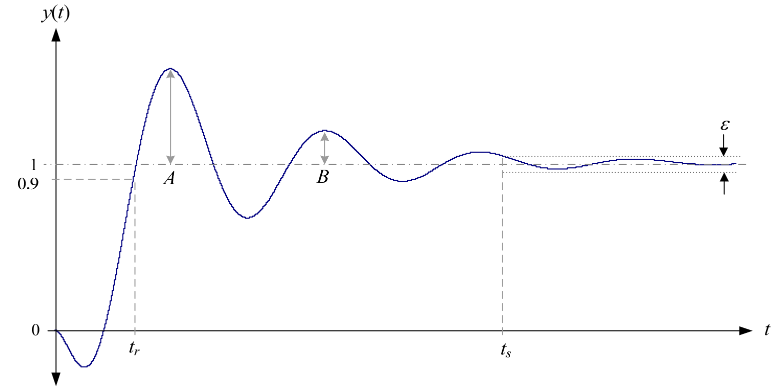
- PID (Proportional, Integral, Derivative) controllers enable intuitive loop tuning in the time domain, normally against a step response. Typical objectives are delay, overshoot, & settling time.

- Most common "parallel" form is $u(t) = k\left(1 + \frac{1}{\tau_i}\int e(t)\,dt + \tau_d \frac{de}{dt}\right)$

- General notes:
  - The proportional term k directly controls loop gain.
  - Integral action increases low frequency gain and reduces/eliminates steady state errors, however this can have a de-stabilizing effect due to increased phase shift.
  - Derivative action introduces phase lead which improves stability and increases bandwidth. This tends to stabilize the loop but can lead to large control movements.

- Many refinements and tuning methods exist (Ziegler-Nichols, Cohen-Coon, etc.)

Transient Response Tuning



<u>Transient response specifications</u>

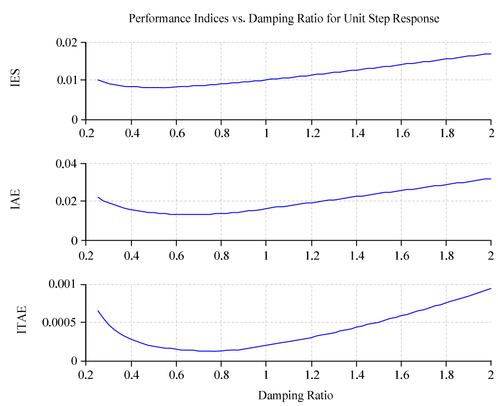
- *Overshoot* (A) - typically 20%
- *Decay ratio* (D) - typically <0.3.
- *Error bound* (ε) - typically 2% of step size
- *Settling time* (t_s) - small
- *Rise time* (t_r) – small
- *Steady state offset* - small

$$D = \frac{B}{A}$$

Tutorial 3.4

Quality of Response

A **performance index** can be used to assess transient response quality against variation of a key parameter.



IES = **Integral of the Error Squared**

$$\int_0^\infty e(t)^2 dt$$

IAE = **Integral of the Absolute Error**

$$\int_0^\infty |e(t)| dt$$

ITAE = **Integral of Time x Absolute Error**

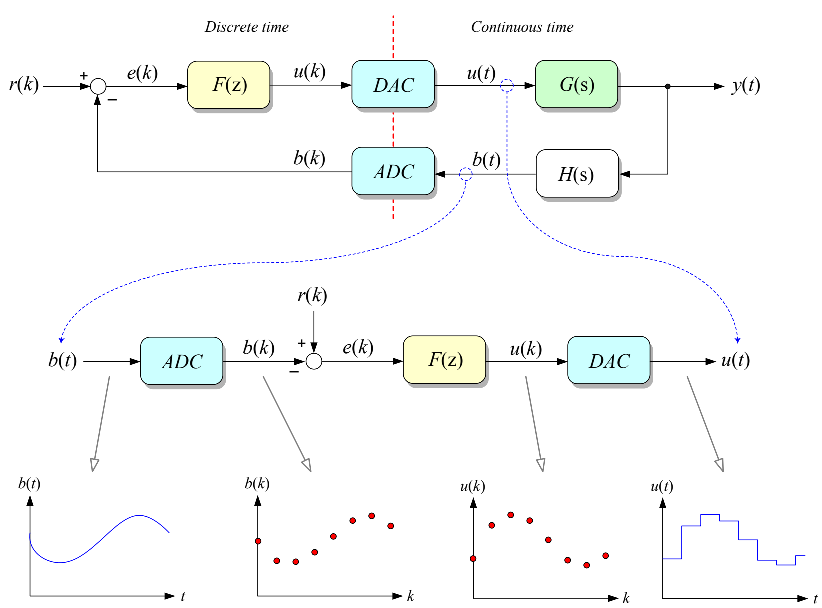
$$\int_0^\infty t\,|e(t)|\,dt$$

For second order systems, a damping ratio of ~0.7 gives a minimum value for IAE & ITAE, and ~0.5 for IES.

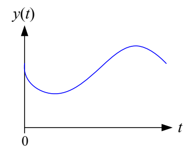
4. Discrete Time Systems

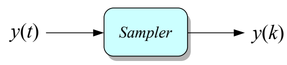
- Sampled systems
- The z-transform
- z plane mapping

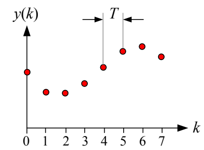
The Digital Control System



Sampled Systems



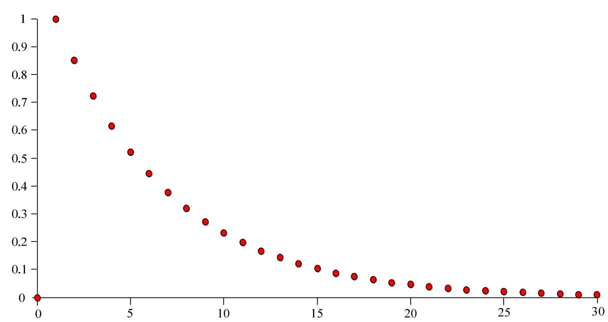




- The sampler converts a continuous function of time $y(t)$ into a discrete time function $y(kT)$
- Almost all samplers operate at a fixed rate $f_s = \dfrac{1}{T}$
- The T is implicit in notation, so for example $y(k)$ is equivalent to $y(kT)$
- The dynamic properties of the signal are changed as it passes through the sampler

Discrete Convolution

The impulse response of a discrete system is its' response to a single input pulse of unit amplitude at time $t = 0$.

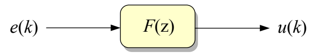


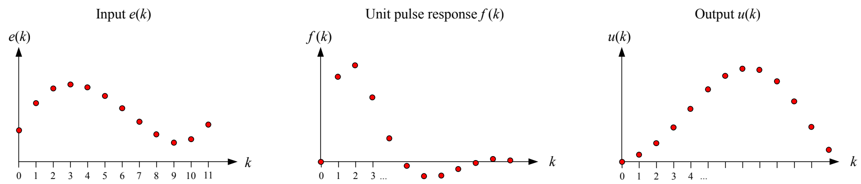
Once the impulse response $f(nT)$ is known, the controller output $u(nT)$ arising from any arbitrary input $e(nT)$ can be computed using the summation

$$u(nT) = \sum_{k=0}^{n} e(kT)\, f([n-k]T)$$

- The design task is to find the $f(nT)$ coefficients which deliver a desired output $u(nT)$ for some $e(nT)$

Discrete Convolution

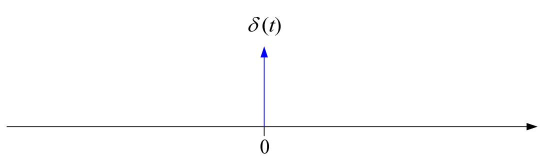




$$
\begin{array}{ll}
T = 0: & u(0) = f(0)e(0) \\
T = 1: & u(1) = f(0)e(1) + f(1)e(0) \\
T = 2: & u(2) = f(0)e(2) + f(1)e(1) + f(2)e(0) \\
T = 3: & u(3) = f(0)e(3) + f(1)e(2) + f(2)e(1) + f(3)e(0) \\
 & \\
T = n: & u(3) = f(0)e(n) + f(1)e(n-1) + \;\ldots\; + f(n)e(0)
\end{array}
$$

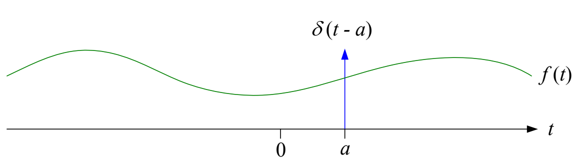
- Discrete convolution consists of sequence reversal, cross-multiplication, & summation
- The digital controller implements this n-term **sum-of-products** at each sample instant, T

The Delta Function

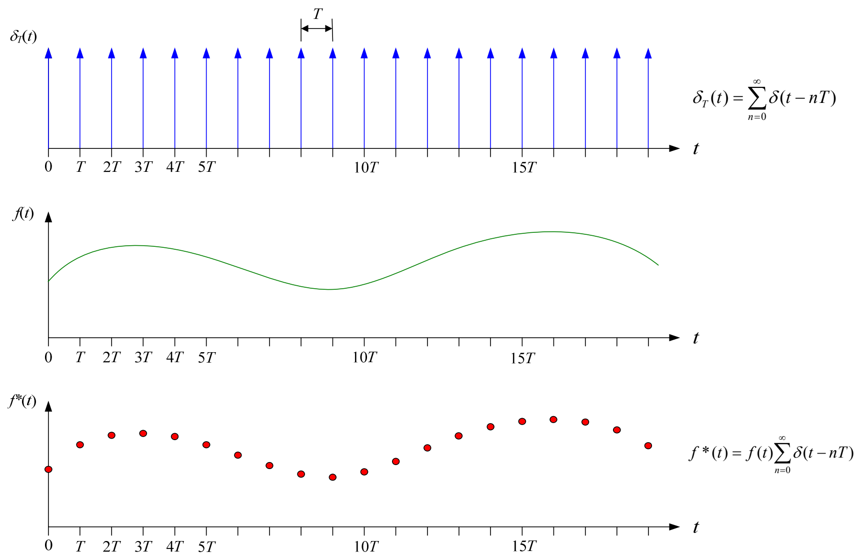


- The delta function, denoted $\delta(t)$, represents an impulse of infinite amplitude, zero width, and unit area.
- If a delta impulse is combined with a continuous signal the result is given by the **screening property**

$$\int_{-\infty}^{\infty} \delta(t-a) f(t)\,dt = f(a)$$



Impulse Modulation



The z Transform

Applying the screening property of the delta function at each sample instant, we find

$$f^*(t) = \sum_{n=0}^{\infty} f(nT)\delta(t - nT)$$

The shifting theorem allows us to take the Laplace transform of this series term-by-term...

$$F^*(s) = \mathscr{L}\left[f(0)\delta(t) + f(T)\delta(t-T) + f(2T)\delta(t-2T) + \ldots\right]$$

$$F^*(s) = \sum_{n=0}^{\infty} f(nT)e^{-snT}$$

The z transform of $f(t)$ is found from the above series after making the substitution $z = e^{sT}$

$$F(z) = \sum_{n=0}^{\infty} f(nT)z^{-n}$$

z Transforms

$f(nT)$	$F(z)$
$\delta(t)$	z^{-n}
$\sum_{n=0}^{\infty}\delta(t-nT)$	$\frac{z}{z-1}$
$1(t)$ (unit step)	$\frac{z}{z-1}$
nT (unit ramp)	$\frac{Tz}{(z-1)^2}$
$(nT)^2$	$\frac{T^2 z(z+1)}{2(z-1)^3}$
e^{-anT}	$\frac{z}{z-e^{-aT}}$

$f(nT)$	$F(z)$
$nT\,e^{-anT}$	$\frac{Tze^{-aT}}{(z-e^{-aT})^2}$
$1-a^{nT}$	$\frac{z(1-a)}{(z-1)(z-a)}$
$1-e^{-at}$	$\frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$
$\sin\omega nT$	$\frac{z\sin\omega T}{z^2-2z\cos\omega T+1}$
$e^{-anT}\sin\omega nT$	$\frac{z e^{-aT}\sin\omega T}{z^2-2z e^{-aT}\cos\omega T+e^{-2aT}}$
$\cos\omega nT$	$\frac{z(z-\cos\omega T)}{z^2-2z\cos\omega T+1}$
$e^{-anT}\cos\omega nT$	$\frac{z^2-z e^{-aT}\cos\omega T}{z^2-2z e^{-aT}\cos\omega T+e^{-2aT}}$

z Transform Theorems

- Convolution of time sequences: $$\mathcal{Z}\left\{\sum_{k=0}^{n} f_1(kT)\,f_2([n-k]T)\right\} = F_1(z)\,F_2(z)$$
- Linearity: $$\mathcal{Z}\,[a_1 f_1(nT) \pm a_2 f_2(nT)] = a_1 F_1(z) \pm a_2 F_2(z)$$
- Time shift: $$\mathcal{Z}\{f(n+k)\} = z^k F(z)$$
- Final value theorem: $$\lim_{n\to\infty} f(nT) = \lim_{z\to 1}(z-1)F(z)$$

z Plane Poles & Zeros

Assume we have a continuous system with the Laplace transfer function:

$$F(s) = \frac{k(s-z_1)(s-z_2)...(s-z_m)}{(s-p_1)(s-p_2)...(s-p_n)}$$

The z-transform of an equivalent sampled data system can be found using a **discrete transformation**, which yields a transfer function in the complex variable z:

$$F(z) = \frac{k'(z-z_1')(z-z_2')...(z-z_m')}{(z-p_1')(z-p_2')...(z-p_n')}$$

- Poles & zeros in the discrete domain are in different positions compared with those in the Laplace domain
- The pole excess of the continuous and discrete representations may not be the same
- Time and frequency domain performance of the two systems will be different

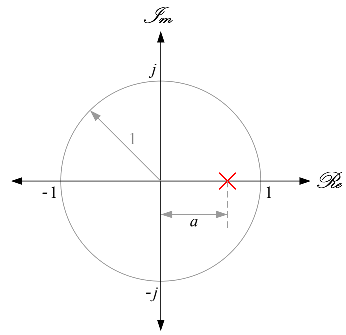
Stability

The first order discrete time transfer function $\frac{u(z)}{e(z)} = \frac{b}{z-a}$

...corresponds to the difference equation $u(k+1) = be(k) + au(k)$

The evolution of the time sequence is:

k	$u(k)$
1	0
2	$be(1)$
3	$be(2) + abe(1)$
4	$be(3) + abe(2) + a^2be(1)$
5	$be(4) + abe(3) + a^2be(2) + a^3be(1)$
n	$b\sum_{i=1}^{n-1} a^i e(n-i)$

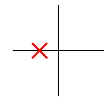


- For stability we have the constraint on the pole: $|a| \leq 1$

Simple z Plane Poles

Recall for continuous-time systems:

$G(s) = \dfrac{1}{s-\mathrm{p}}$ has an impulse response of the form $y(t) = e^{\mathrm{p}t}$



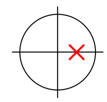
If $\mathrm{p} = \alpha + j\beta$, $u(t) = e^{\alpha t} e^{j\beta t}$, and for stability we want $e^{\alpha t}$ to remain finite.

$i.e.$ the pole at $(s = \mathrm{p})$ must have a negative real part, and

- ***stable poles must lie in the left half of the s-plane***

Now, in discrete systems:

$G(\mathrm{z}) = \dfrac{z}{z-\mathrm{p}}$ has a unit pulse response $y(nT)$ involving the term p^n



Therefore, for stability $|\mathrm{p}|$ must be less than 1, and

- ***stable poles must lie within the unit circle in the z-plane***

Complex z Plane Poles

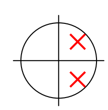
As for continuous time systems, discrete time complex poles always exist as conjugate pairs.

$$G(z) = \frac{z^2}{(z-pe^{j\omega})(z-pe^{-j\omega})}$$

The impulse response is given by $y(z) = \dfrac{A_1}{(z-pe^{j\omega})} + \dfrac{A_1^*}{(z-pe^{-j\omega})}$

$$y(nT) = A_1\left(pe^{j\omega}\right)^n + A_1^*\left(pe^{-j\omega}\right)^n$$

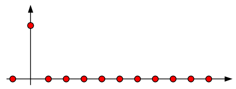
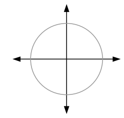
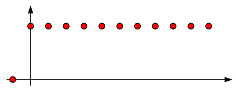
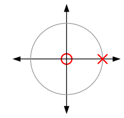
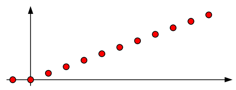
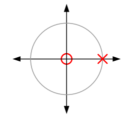
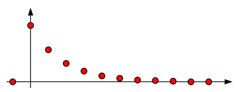
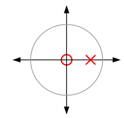
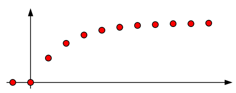
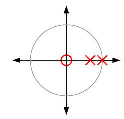
...where the residual A_1 has the form $ae^{j\theta}$



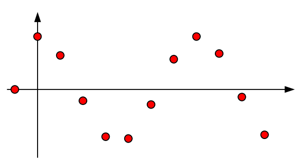
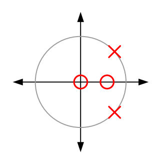
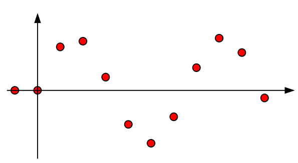
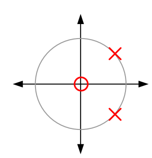
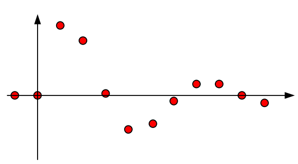
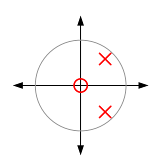
The time sequence can be written $y(nT) = 2\alpha p^n \cos(n\omega + \theta)$

- ***complex poles give rise to an oscillatory response and have the same stability constraints as simple poles***

z Transforms of Common Signals

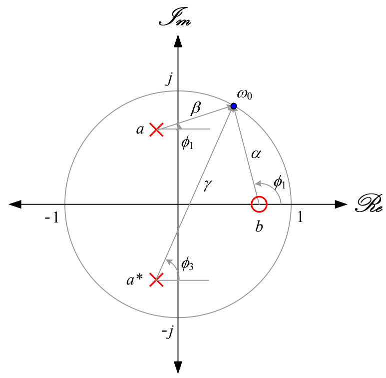
Waveform	$f(nT)$	$F(z)$	z-plane
	$\delta[T]$	1	
	1	$\dfrac{z}{z-1}$	
	nT	$\dfrac{z}{(z-1)^2}$	
	a^n	$\dfrac{z}{z-a}$	
	$1-a^n$	$\dfrac{z(1-a)}{(z-a)(z-1)}$	

z Transforms of Common Signals

Waveform	$f(nT)$	$F(z)$	z-plane
	$\cos anT$	$\dfrac{z(z-\cos aT)}{z^2-2z\cos aT+1}$	
	$\sin anT$	$\dfrac{z\sin aT}{z^2-2z\cos aT+1}$	
	$a^n \sin bnT$	$\dfrac{az\sin bT}{z^2-2az\cos bT+a^2}$	

Frequency Response

- The response of $G(j\omega)$ at frequency $\omega = \omega_0$ is evaluated by $G(z)\big|_{z=e^{j\omega_0 T}}$

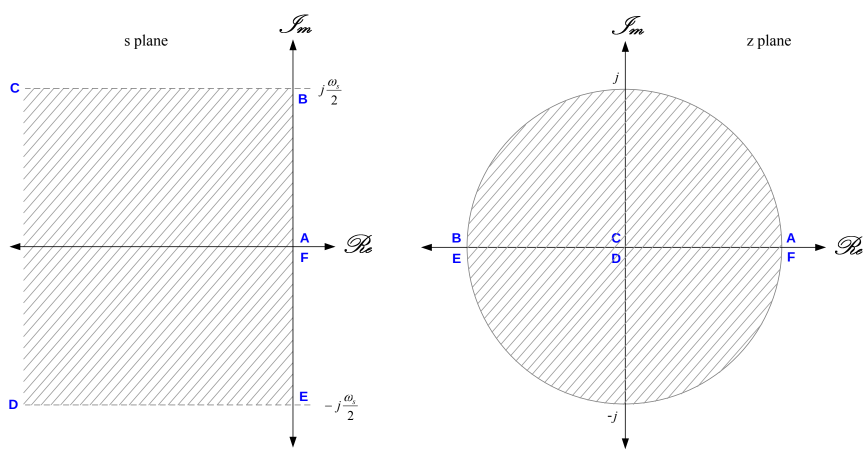


For the first order system $G(z) = \dfrac{z+b}{(z+a)(z+a^*)}$ magnitude and phase are found from

$$\left|G(e^{j\omega_0 T})\right| = \frac{\left|e^{j\omega_0 T} + b\right|}{\left|e^{j\omega_0 T} + a\right|\left|e^{j\omega_0 T} + a^*\right|} = \frac{\alpha}{\beta\gamma}$$

$$\angle G(e^{j\omega_0 T}) = \angle\left(e^{j\omega_0 T} + b\right) - \angle\left(e^{j\omega_0 T} + a\right) - \angle\left(e^{j\omega_0 T} + a^*\right) = \phi_1 - \phi_2 - \phi_3$$

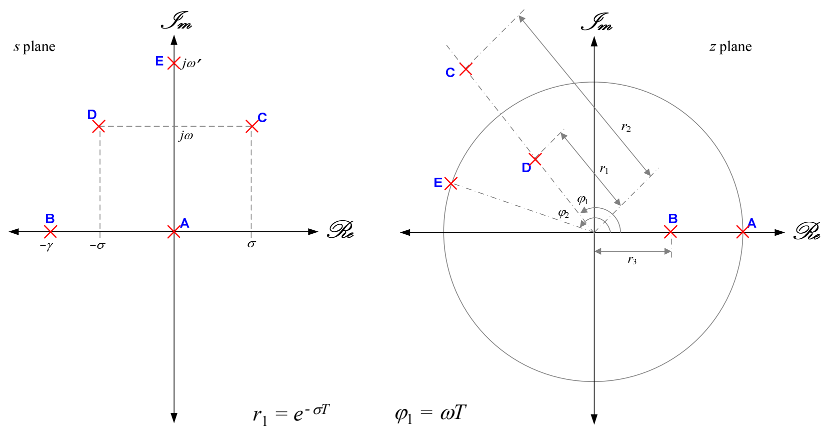
z Plane Mapping



Equivalent regions shown cross-hatched

Complex Plane Mapping

- Points in the s-plane are mapped according to $z = e^{sT} = e^{(a+jb)T} = e^{aT}e^{jbT} = re^{j\varphi}$



$$r_1 = e^{-\sigma T} \qquad \varphi_1 = \omega T$$

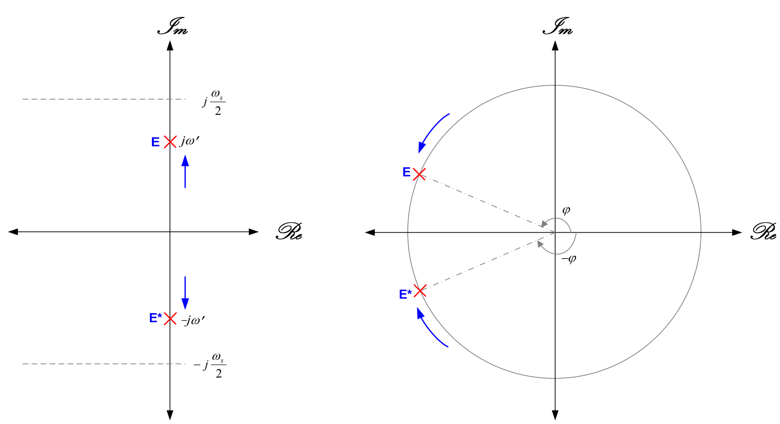
$$r_2 = e^{\sigma T} \qquad \varphi_2 = \omega' T$$

$$r_3 = e^{-\gamma T}$$

The Nyquist Frequency

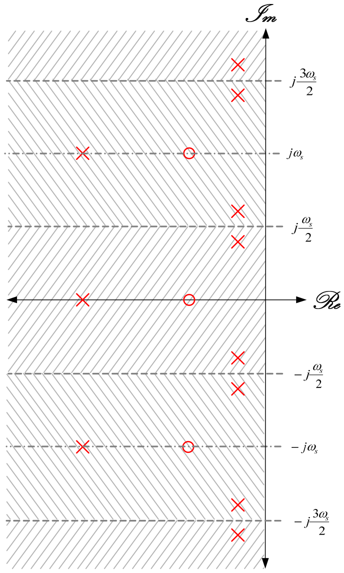
s-plane

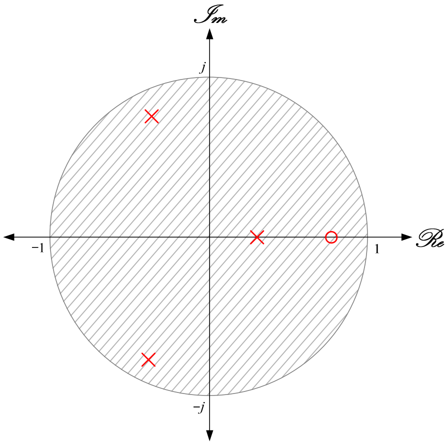
z-plane



- The Nyquist frequency represents the highest unique frequency in the discrete time system
- Uniqueness is lost for higher continuous time frequencies after sampling

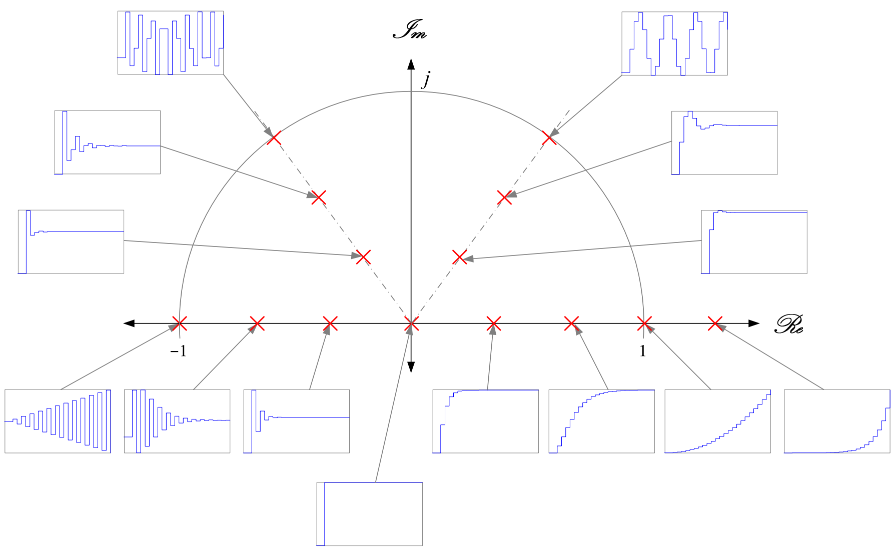
Aliasing





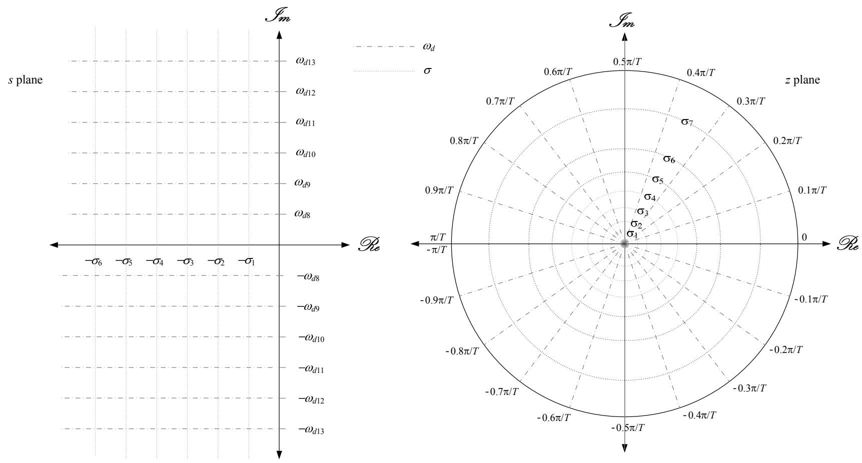
- Loss of uniqueness means an infinite number of congruent strips are mapped into the unit circle

Pole Location vs. Step Response



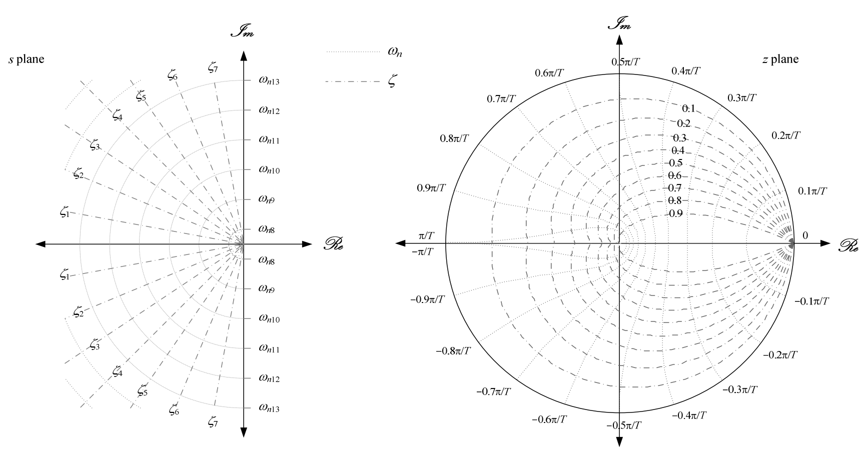
Complex Plane Grid

Lines of constant decay rate (σ) and damped natural frequency (ω_d)



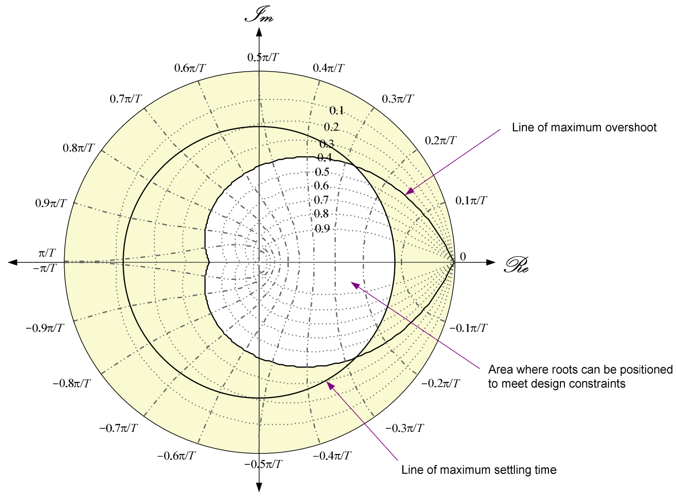
Complex Plane Grid

Lines of constant damping ratio (ζ) and un-damped natural frequency (ω_n)



Root Locus Design Constraints

- Equivalent second order loci allow regions of the complex plane to be marked out which correspond to closed loop root locations yielding acceptable transient response

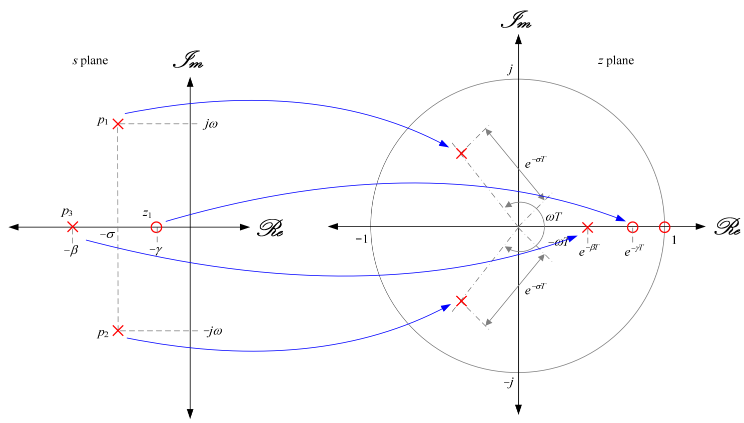


5. Digital Control Design

- **Pole-zero matching**
- **Numerical approximation**
- **Invariant methods**
- **Direct digital design**

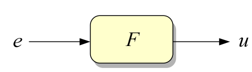
Pole – Zero Matching

1. Transform the poles & zeros of the transfer function using $z = e^{sT}$
2. Map any infinite zeros to $z = -1$ (but maintain a pole excess of 1)
3. Match the gain of the transformed system at $z = 1$ to that of the original at $s = 0$



$$F(s) = A\frac{(s+z_1)}{(s+p_1)(s+p_2)(s+p_2)} \Rightarrow F(z) = A_1\frac{(z+1)(z-e^{-z_1T})}{(z-e^{-p_1T})(z-e^{-p_2T})(z-e^{-p_3T})}$$

Numerical Approximation

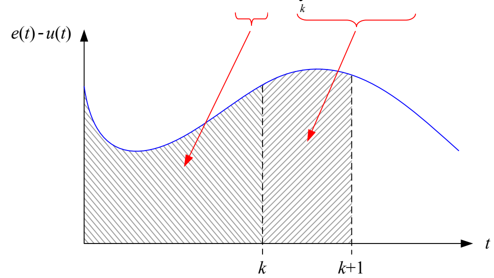


Starting with the simple controller $F(s) = \dfrac{a}{s+a}$ we get the differential equation $\dot{u}(t) + au(t) = ae(t)$

The solution to the continuous equation is $u(t) = a\int_0^t (e(t) - u(t))\,dt$

An equivalent discrete time controller performs this integration in discrete time:

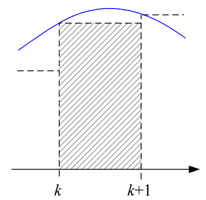
$$u(k+1) = u(k) + a\int_k^{k+1} (e(t) - u(t))\,dt$$



Forward Approximation Method

Approximating the unknown area using a rectangle of height $a\{e(k)\ -\ u(k)\}$...

$$u(k+1) = u(k) + aT\left[e(k) - u(k)\right]$$

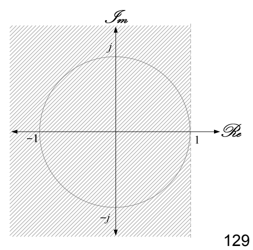


Application of the shifting theorem and simple algebra leads to...

$$F(z) = \frac{u(z)}{e(z)} = \frac{a}{\left(\frac{z-1}{T}\right) + a}$$

The forward approximation method implies we can find the z-transform directly from the Laplace transform by making the substitution:

$$s \leftarrow \frac{z-1}{T}$$

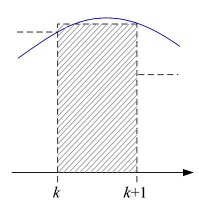


The forward approximation rule maps the ROC of the s plane into the region shown. The unit circle is a subset of the mapped region, so stability is not necessarily preserved under this mapping.

Backward Approximation Method

Approximating the unknown area using a rectangle of height $a\{e(k+1)\ -\ u(k+1)\}$...

$$u(k+1) = u(k) + aT\left[e(k+1) - u(k+1)\right]$$

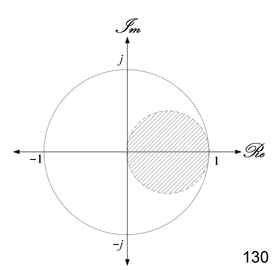


Application of the shifting theorem and simple algebra leads to...

$$F(z) = \frac{a}{\left(\frac{z-1}{Tz}\right) + a}$$

The backward approximation method implies we can find the z-transform directly from the Laplace transform by making the substitution:

$$s \leftarrow \frac{z-1}{Tz}$$

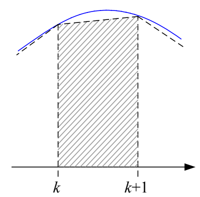


The backward approximation rule maps the ROC of the s plane into a circle of radius 0.5 within the z plane unit circle. Pole-zero locations are very distorted under this mapping.

Trapezoidal Approximation Method

Approximating the unknown area using a trapezoid...

$$u(k+1) = u(k) + \frac{aT}{2}\left[e(k) - u(k) + e(k+1) - u(k+1)\right]$$

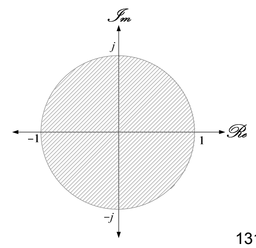


Application of the shifting theorem and simple algebra leads to...

$$F(z) = \frac{a}{\left(\frac{2}{T}\frac{z-1}{z+1}\right) + a}$$

The trapezoidal approximation method implies we can find the z-transform directly from the Laplace transform by making the substitution:

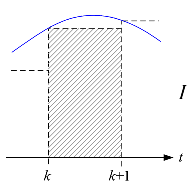
$$s \leftarrow \frac{2}{T}\frac{z-1}{z+1}$$



Trapezoidal approximation maps the ROC of the s plane exactly into the unit circle.

This method is also known as **Tustin's method** or the **bi-linear transform**.

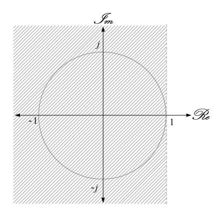
Numerical Approximation Methods

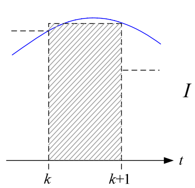


Forward approximation

$$I = aT\left[e(k) - u(k)\right]$$

$$s \leftarrow \frac{z-1}{T}$$

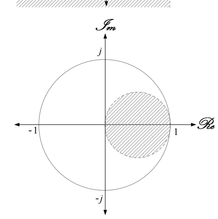


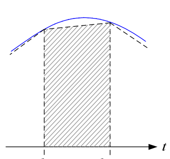


Backward approximation

$$I = aT\left[e(k+1) - u(k+1)\right]$$

$$s \leftarrow \frac{z-1}{Tz}$$

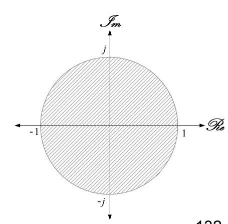




Trapezoidal approximation

$$I = a\frac{T}{2}\left[e(k+1) - u(k+1) - e(k) + u(k)\right]$$

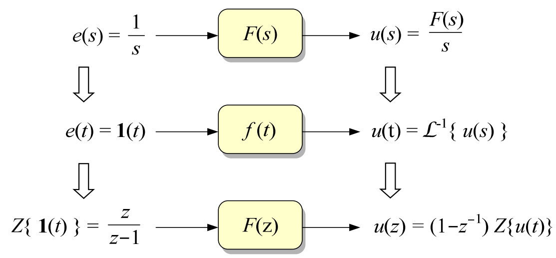
$$s \leftarrow \frac{2}{T}\frac{z-1}{z+1}$$



Step Invariant Method

- Invariant methods emulate the response of the continuous system to a specific input

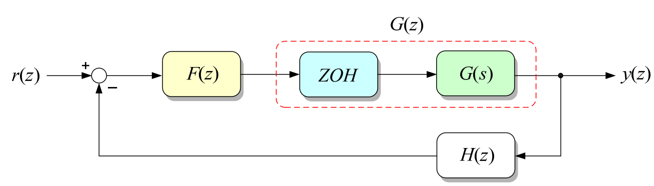
  1. Determine the output of the output of the continuous time system for the selected hold input
  2. Find the corresponding z-transform of the response
  3. Divide by the z-transform of the selected input



- Step and ramp invariant methods are also known as ZOH and FOH equivalent methods respectively

- Invariant methods capture the gain & phase characteristics of the respective hold unit

Direct Digital Design



- In direct design we begin by transforming the plant model into discrete form using the step invariant method. This captures the action of the ZOH element which precedes the plant.

- Standard design techniques can then be used to synthesize the controller.

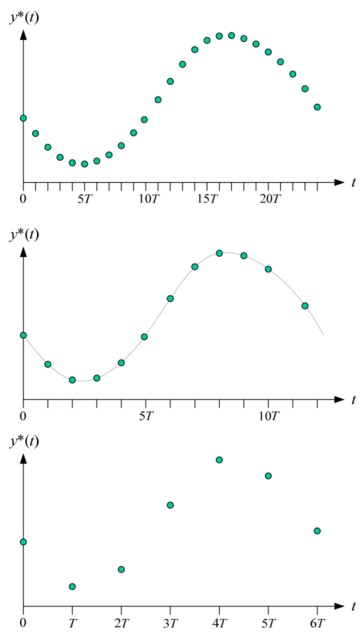
- The design iterates as many times as necessary until a satisfactory controller is reached.

- Control performance with the direct method is usually significantly better than with emulation methods.

6. Digital Control System Implementation

- Sample rate selection
- Sample to output delay
- Reconstruction
- Control law implementation
- Aliasing

Sample Rate Selection

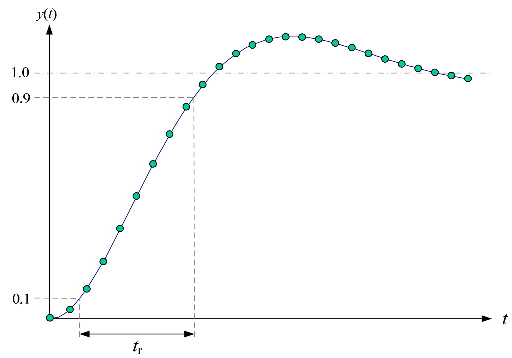


Fast sample rate
- Increased CPU load
- More expensive processor
- More expensive data converters
- Faster transient response
- Smaller & cheaper passive components
- Reduced output ripple
- Better disturbance rejection
- Simpler & cheaper anti-aliasing filters

Slow sample rate
- Reduced CPU load
- Cheaper processor
- Cheaper data converters
- Performance trade-offs required
- Larger passives components required
- Increased output ripple
- More sensitive to disturbances
- More complex & expensive anti-aliasing filters

Sample Rate Selection

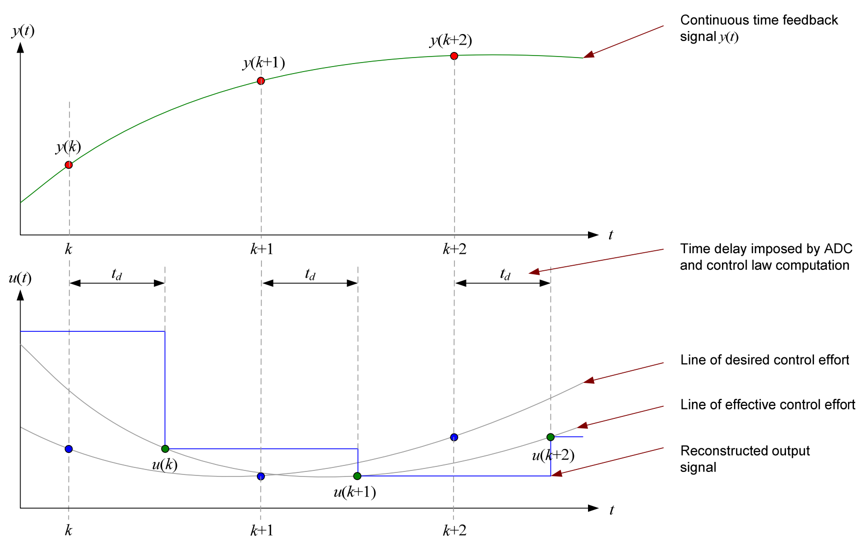
- For control systems, choice of sample rate is based on the 10% to 90% rise time of the plant



Define N as the ratio of rise time to sample period: $N = \dfrac{t_r}{T}$ or $T = \dfrac{t_r}{N}$

- Acceptable results will normally be obtained for the range $4 \leq N \leq 10$

Sample to Output Delay

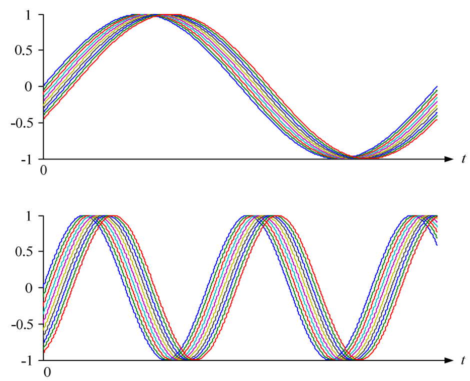


t_d = sample to output delay

Phase Delay

From the shifting property of the Laplace transform, if $\mathscr{L}\{f(t)\} = F(s)$ then $\mathscr{L}\{f(t-\phi)\} = e^{-s\phi}F(s)$

Therefore, if $F(j\omega) = re^{j\theta}$ then $e^{-j\omega D}F(j\omega) = re^{j(\theta - \omega D)}$



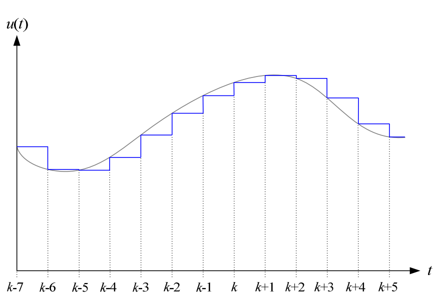
- Phase lag is indistinguishable from delay in the time domain.
- Delay in the time domain translates into frequency dependent phase lag in the frequency domain

Reconstruction

Hold functions attempt to construct a smooth continuous time signal from a discrete time sequence.

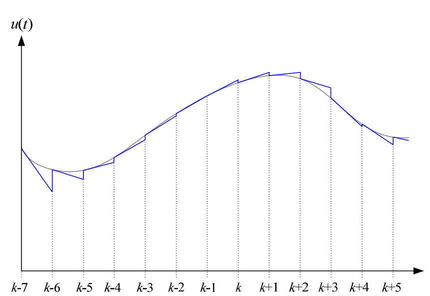
To do so, the function must approximate over the interval $kT \le t \le (k+1)T$

Zero Order Hold (ZOH)



$$u(t) = u(k)$$

First Order Hold (FOH)



$$u(t) = u(k) + \frac{u(k) - u(k-1)}{T}[t - kT]$$

Zero Order Hold

The frequency response of the Zero Order Hold function can be modelled using a unit pulse

$$F_{ZOH}(j\omega) = \frac{1 - e^{-j\omega T}}{j\omega}$$

This can be simplified using the exponential form of the sine function

$$F_{ZOH}(j\omega) = e^{-j\omega\frac{T}{2}} \left\{ \frac{e^{j\omega\frac{T}{2}} - e^{-j\omega\frac{T}{2}}}{2j} \right\} \frac{2j}{j\omega} = \frac{2}{\omega} \sin\left(\frac{\omega T}{2}\right) e^{-j\omega\frac{T}{2}}$$

Further simplification using the sinc function yields $F_{ZOH}(j\omega) = T \operatorname{sinc}\left(\frac{\omega T}{2}\right) e^{-j\omega T/2}$

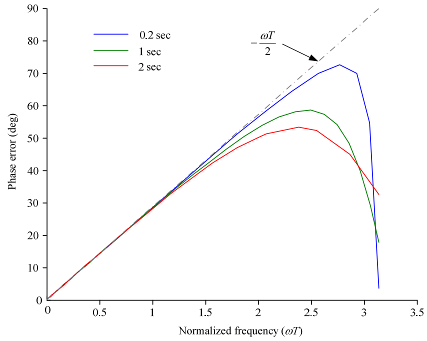
This is a complex number expressed in polar form, where the angle is given by

$$\boxed{\angle F_{ZOH}(j\omega) = -\omega \frac{T}{2} = -\frac{\omega}{\omega_s}}$$

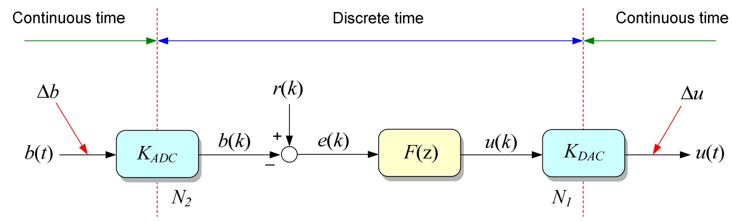
- The Zero Order Hold contributes a frequency dependent phase lag to the loop response

ZOH Phase Approximation

- Comparison of phase error vs. normalised frequency for various sample rates shows the approximation $\phi = -\omega \frac{T}{2}$ holds reasonably well for $\omega < \frac{\omega_s}{4}$



Data Converter Gain



- Quantization takes place at each continuous/discrete boundary. Resolution limits can be referred to the continuous domain by:

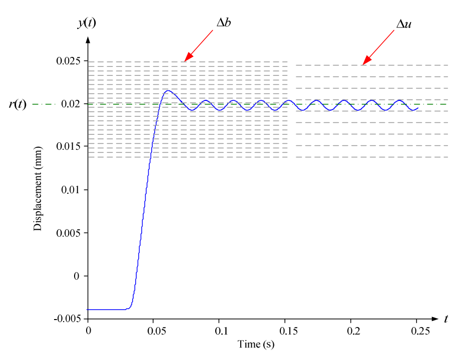
$$\Delta b = \frac{b_{\max}}{2^{N_2}} \qquad \Delta u = \frac{u_{\max}}{2^{N_1}}$$

- Converter gains are related to quantization limits by:

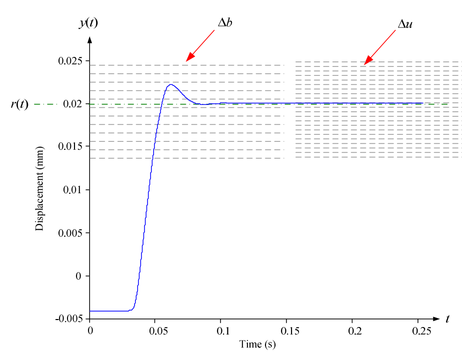
$$K_{ADC} = \frac{1}{\Delta b} \qquad K_{DAC} = \Delta u$$

- End-to-end controller gain is $K_{ADC} \mid F(z) \mid K_{DAC}$

Resolution Based Limit Cycles



$\Delta u > \Delta b$



$\Delta u < \Delta b$

- Steady-state limit cycles may appear if the sense resolution (Δb) is worse than the control resolution (Δu)

- Limit cycle oscillation will be low amplitude & high frequency, and may not cause an issue in low bandwidth systems, or in systems with high inertia or friction. In some cases, the oscillation may even be beneficial as it introduces a “dither-like” effect.

The Limit Cycle Condition

The "limit cycle condition" is: control resolution > sense resolution: $\Delta u > \Delta b$

Control resolution is: $\Delta u = \dfrac{u_{\max}}{2^{N_1}}$

Sense resolution is: $\Delta b = \dfrac{b_{\max}}{2^{N_2}}$

In the steady state, control and sense quantities are related by $b_{\max} = |G(0)H(0)|\, u_{\max} = k\, u_{\max}$

This gives the inequality $\dfrac{u_{\max}}{2^{N_1}} < k \dfrac{u_{\max}}{2^{N_2}}$

$$2^{N_1} > \frac{1}{k} 2^{N_2}$$

- To avoid a steady-state limit cycle, control effort resolution must satisfy the inequality

$$\boxed{N_1 > N_2 - \log_2 k}$$

Finding the Difference Equation

The n-pole m-zero transfer function is written

$$\frac{u(z)}{e(z)} = \frac{\beta_0 z^m + \beta_1 z^{m-1} + \ldots + \beta_{m-1} z + \beta_m}{\alpha_0 z^n + \alpha_1 z^{n-1} + \ldots + \alpha_{n-1} z + \alpha_n}$$

Normalizing for the term involving the highest denominator power (a_0) gives

$$\frac{u(z)}{e(z)} = \frac{b_0 z^{m-n} + b_1 z^{m-1-n} + \ldots + b_{m-1} z^{1-n} + b_m z^{-n}}{1 + a_1 z^{-1} + \ldots + a_{n-1} z^{1-n} + a_n z^{-n}}$$

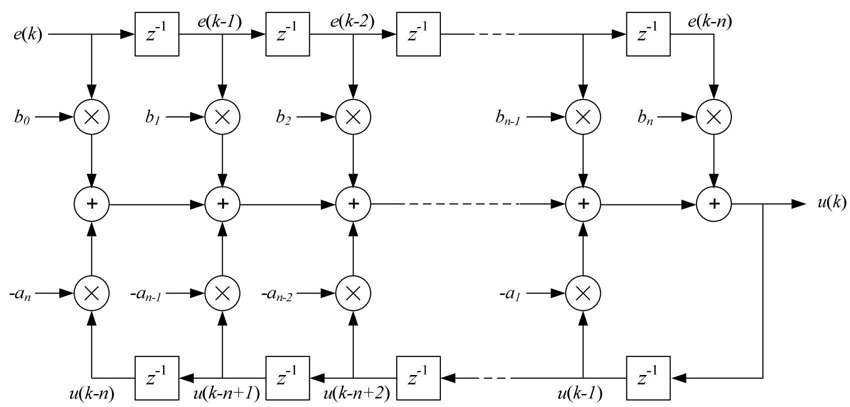
$$u(z) = e(z)\left\{b_0 z^{m-n} + b_1 z^{m-1-n} + \ldots + b_{m-1} z^{1-n} + b_m z^{-n}\right\} - u(z)\left\{a_1 z^{-1} + \ldots + a_{n-1} z^{1-n} + a_n z^{-n}\right\}$$

Applying the shifting property of the z-transform term-by-term yields the difference equation

$$\boxed{u(k) = b_0 e(k-n+m) + b_1 e(k-n+m+1) + \ldots + b_{m-1} e(k-n+1) + b_m e(k-n) - a_1 u(k-1) - \ldots - a_{n-1} u(k-2) - a_n u(k-n)}$$

Control Law Diagram

The general n-pole n-zero control law can be represented in diagrammatic form as shown below...

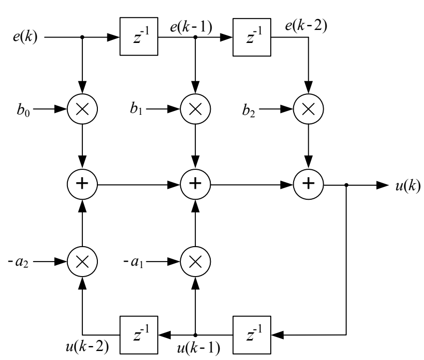


$$u(k) = b_0 e(k) + b_1 e(k-1) + b_2 e(k-2) + \ldots + b_n e(k-n) - a_1 u(k-1) - \ldots - a_{n-1} u(k-n+1) - a_n u(k-n)$$

2P2Z Control Law

$$u(k) = b_0 e(k) + b_1 e(k-1) + b_2 e(k-2) - a_1 u(k-1) - a_2 u(k-2)$$

The 2-pole 2-zero (2P2Z) control law can be represented graphically as shown below.

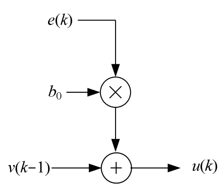


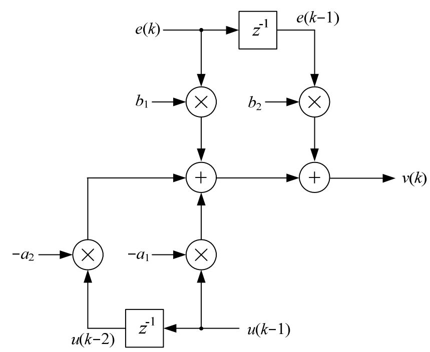
Pre-Computation

Sample-to-output delay can be reduced by pre-computing that part of the control law which is already available.

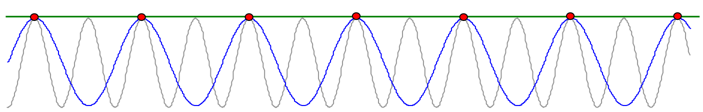
$$u(k) = \underbrace{b_0 e(k)}_{\text{Unknown at } t=k-1} + \underbrace{b_1 e(k-1) + b_2 e(k-2) - a_1 u(k-1) - a_2 u(k-2)}_{\text{Can be pre-computed at } t=k-1}$$

$$u(k) = b_0 e(k) + v(k-1) \qquad\qquad v(k) = b_1 e(k) + b_2 e(k-1) - a_1 u(k-1) - a_2 u(k-2)$$





Series of Constant Samples



The sampler modifies the frequency response of a signal according to:

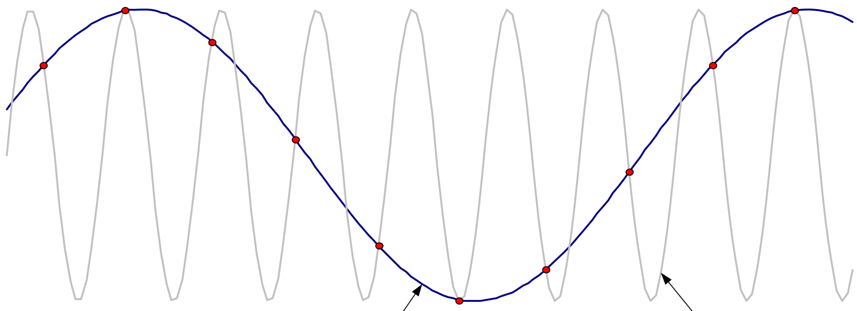
$$e^{sT}\Big|_{s=j\omega} = e^{j\omega T} = \cos\omega T + j\sin\omega T$$

Since $T = \dfrac{2\pi}{\omega_s}$, this can be written $e^{j2\pi\frac{\omega}{\omega_s}} = \cos\left(2\pi\dfrac{\omega}{\omega_s}\right) + j\sin\left(2\pi\dfrac{\omega}{\omega_s}\right)$

When $\omega = n\omega_s$ ... $\qquad e^{j2\pi\frac{\omega}{\omega_s}}\Big|_{\omega=n\omega_s} = \cos(2n\pi) + j\sin(2n\pi) = 1 \qquad$ (n integer)

- Whenever the input frequency is a multiple of the sample rate ($\omega = n\omega_s$), a series of constant sample values will be produced. *i.e.* DC is indistinguishable from any signal which is a multiple of the sample rate.

Discrete Frequency Ambiguity



- Both f_1 & f_2 give rise to exactly the same set of samples. After sampling it is impossible to determine which frequency was sampled. In fact, any of an infinite number of possible sine waves could have produced these samples. This effect is known as **aliasing**.

Frequency Response of a Sampled System

The sampled signal is given by $y^*(t) = \sum_{k=-\infty}^{\infty} y(t)\delta(t-kT)$

The sampler is periodic so can be represented by the Fourier series $\sum_{k=-\infty}^{\infty} \delta(t-kT) = \sum_{n=-\infty}^{\infty} C_n\, e^{jn\omega_s t}$

...where the Fourier coefficients are given by $C_n = \frac{1}{T}\int_{-T/2}^{T/2} \sum_{k=-\infty}^{\infty} \delta(t-kT)\; e^{-jn\omega_s t}\, dt$

Only one term is within range of the integration, so $C_n = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, e^{-jn\omega_s t}\, dt$

We can integrate this easily using the “screening” property of the delta function $\int_{-\infty}^{\infty} f(t)\,\delta(t-a)\,dt = f(a)$

$$C_n = \frac{1}{T}\left[e^0\right]_{-T/2}^{T/2} = \frac{1}{T}$$

So, the Fourier series representing the sampler is given by $\sum_{k=-\infty}^{\infty} \delta(t-kT) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_s t}$

Frequency Response of a Sampled System

$$\sum_{k=-\infty}^{\infty}\delta(t-kT)=\frac{1}{T}\sum_{n=-\infty}^{\infty}e^{jn\omega_s t}$$

We can now find the Laplace transform of the sampled system $\mathscr{L}\{f(t)\}=\int_0^{\infty} f(t)e^{-st}dt = F(s)$

$$Y^*(s)=\mathscr{L}\{y^*(t)\}=\int_0^{\infty} y(t)\left\{\frac{1}{T}\sum_{n=-\infty}^{\infty}e^{jn\omega_s t}\right\}e^{-st}dt$$

$$Y^*(s)=\frac{1}{T}\sum_{n=-\infty}^{\infty}\int_0^{\infty} y(t)e^{-(s-jn\omega_s)t}dt$$

The integral term is the same as the Laplace transform of $y(t)$, but with a change of complex variable

$$Y^*(s)=\frac{1}{T}\sum_{n=-\infty}^{\infty}Y(s-jn\omega_s)$$

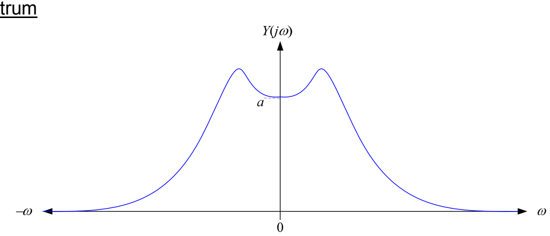
The frequency response of the samples signal is:

$$Y^*(j\omega)=\frac{1}{T}\sum_{n=-\infty}^{\infty}Y(j[\omega-n\omega_s])$$

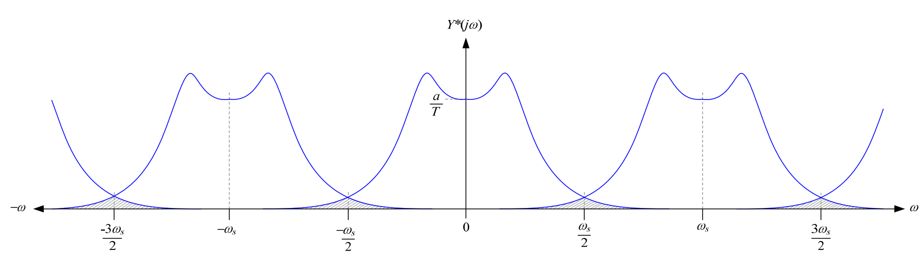
Each term in the infinite summation corresponds to the response of the continuous system, shifted along the frequency axis by $\pm n\omega_s$

Frequency Response of a Sampled System

Continuous Spectrum

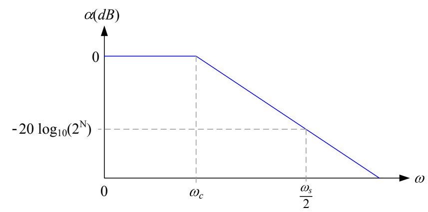


Sampled Spectrum

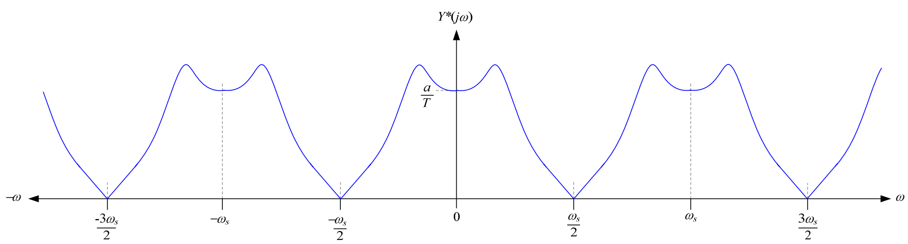


Anti-Aliasing

- To prevent aliasing, we need to attenuate the input signal to less than 1 converter bit at $\frac{\omega_s}{2}$ before sampling.



Filter constraints can be relaxed if a faster sample rate is selected.



Suggested Design Checklist

- ✓ **Carefully select a sample rate**
- ✓ **Account for data converter gains**
- ✓ **Design controller using emulation or direct method**
- ✓ **Design anti-aliasing filter**
- ✓ **Select a suitable processor**
- ✓ **Evaluate sample-to-output delay**
- ✓ **Account for ZOH effects**
- ✓ **Simulate & iterate!**

Review

1. **Fundamental Concepts**
   Linear systems
   Transient response classification
   Frequency domain descriptions

2. **Feedback Control**
   Effects of feedback
   Steady state error
   Stability & bandwidth

3. **Controller Design**
   Phase compensation
   Root locus analysis
   Transient tuning

4. **Discrete Time Systems**
   Sampled systems
   The z-transform
   z plane mapping

5. **Digital Control Design**
   Pole-zero matching
   Numerical approximation
   Invariant methods
   Direct digital design

6. **Digital Control Systems Implementation**
   Sample rate selection
   Sample to output delay
   Reconstruction
   Control law implementation
   Aliasing

Suggested Reading

- **J.J.DiStefano, A.R.Stubberud & I.J.Williams, *Feedback & Control Systems*, Schaum, 1995**
- **J.Doyle, B.Francis & A.Tannenbaum, *Feedback Control Theory*, Macmillan, 1990**
- **S. Skogestad & I. Postlethwaite, *Multivariable Feedback Control*, Wiley, 2005**
- **Gene F. Franklin, J. David Powell & Michael L. Workman, *Digital Control of Dynamic Systems*, Addison-Wesley, 1998**
- **K.J.Astrom & B.Wittenmark, *Computer Controlled Systems*, Prentice Hall, 1997**

Richard Poley
Texas Instruments Inc.
r-poley@ti.com

Table of selected Laplace transforms

$f(t)$		$F(s)$
$\delta(t)$		1
$\delta(t-T)$		e^{-sT}
$\mathbf{1}(t)$	(unit step)	$\frac{1}{s}$
$\mathbf{1}(t-T)$	(delayed step)	$\frac{1}{s}e^{-sT}$
$\mathbf{1}(t)-\mathbf{1}(t-T)$	(rectangular pulse)	$\frac{1}{s}(1-e^{-sT})$
t	(unit ramp)	$\frac{1}{s^2}$
$\frac{t^{n-1}}{(n-1)!}$		$\frac{1}{s^n}$
e^{-at}		$\frac{1}{s+a}$
$\frac{1}{(n-1)!}t^{n-1}e^{-at}$		$\frac{1}{(s+a)^n}$
$\sin\omega t$		$\frac{\omega}{s^2+\omega^2}$
$\cos\omega t$		$\frac{s}{s^2+\omega^2}$
$\frac{1}{\omega}e^{-at}\sin\omega t$		$\frac{1}{(s+a)^2+\omega^2}$
$e^{-at}\cos\omega t$		$\frac{s+a}{(s+a)^2+\omega^2}$

$f(t)$	$F(s)$
$\frac{1}{b-a}(e^{-at}-e^{-bt})$	$\frac{1}{(s+a)(s+b)}$
$\frac{1}{a-b}(ae^{-at}-be^{-bt})$	$\frac{s}{(s+a)(s+b)}$
$\frac{1}{a}(1-e^{-at})$	$\frac{1}{s(s+a)}$
$\frac{1}{a^2}(1-e^{-at}-ate^{-at})$	$\frac{1}{s(s+a)^2}$
$\frac{1}{a^2}(at-1-e^{-at}+e^{-at})$	$\frac{1}{s^2(s+a)}$
$\frac{1}{ab}\left(1-\frac{be^{-at}}{b-a}+\frac{ae^{-bt}}{b-a}\right)$	$\frac{1}{s(s+a)(s+b)}$
$\frac{1}{\omega^2}(1-\cos\omega t)$	$\frac{1}{s(s^2+\omega^2)}$
$\frac{1}{\omega_d}e^{-\zeta\omega_n t}\sin\omega_d t$	$\frac{1}{s^2+2\zeta\omega_n s+\omega_n{}^2}$
$\frac{1}{\omega_n^2}-\frac{1}{\omega_n\omega_d}e^{-\zeta\omega_n t}\sin(\omega_d t+\phi)$	$\frac{1}{s(s^2+2\zeta\omega_n s+\omega_n{}^2)}$
$\sin(\omega t+\phi)$	$\frac{s\sin\phi+\omega\cos\phi}{s^2+\omega^2}$

Table of selected z-transforms

$f(nT)$		$F(z)$
$\delta(t)$		z^{-n}
$\sum_{n=0}^{\infty} \delta(t-nT)$		$\dfrac{z}{z-1}$
$1(t)$	(unit step)	$\dfrac{z}{z-1}$
nT	(unit ramp)	$\dfrac{Tz}{(z-1)^2}$
$(nT)^2$		$\dfrac{T^2 z(z+1)}{2(z-1)^3}$
e^{-anT}		$\dfrac{z}{z-e^{-aT}}$
$nT\,e^{-anT}$		$\dfrac{Tze^{-aT}}{(z-e^{-aT})^2}$
$1-a^{nT}$		$\dfrac{z(1-a)}{(z-1)(z-a)}$
$1-e^{-at}$		$\dfrac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$
$\sin \omega nT$		$\dfrac{z \sin \omega T}{z^2 - 2z\cos \omega T + 1}$
$e^{-anT} \sin \omega nT$		$\dfrac{z e^{-aT} \sin \omega T}{z^2 - 2z e^{-aT} \cos \omega T + e^{-2aT}}$
$\cos \omega nT$		$\dfrac{z(z-\cos \omega T)}{z^2 - 2z\cos \omega T + 1}$
$e^{-anT} \cos \omega nT$		$\dfrac{z^2 - z e^{-aT} \cos \omega T}{z^2 - 2z e^{-aT} \cos \omega T + e^{-2aT}}$

Notes

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Thailand	001-800-886-0010

Fax	+886-2-2378-6808
Email	tiasia@ti.com ti-china@ti.com
Internet	support.ti.com/sc/pic/asia.htm